Journal of Statistical Physics, Vol. 64, Nos. 1/2, 1991

Analytic Calculation of Scaling Dimensions: Tricritical Hard Squares and Critical Hard Hexagons

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Received November 28, 1990

The finite-size corrections, central charges c, and scaling dimensions x of tricritical hard squares and critical hard hexagons are calculated analytically. This is achieved by solving the special functional equation or inversion identity satisfied by the commuting row transfer matrices of these lattice models at criticality. The results are expressed in terms of Rogers dilogarithms. For tricritical hard squares we obtain $c=7/10$, $x=3/40$, $1/5$, $7/8$, $6/5$ and for hard hexagons we obtain $c=4/5$, $x=2/15$, $4/5$, $17/15$, $4/3$, $9/5$, in accord with the predictions of conformal and modular invariance.

KEY WORDS: Scaling dimensions; finite-size corrections; conformal invariance; hard squares; hard hexagons.

1. INTRODUCTION

In statistical mechanics it is well established that two-dimensional critical lattice models exhibit scale invariance,[1] conformal invariance,[2,3] and modular invariance.[4,5] Critical behaviors are classified into universality classes according to the central charge c of the Virasoro algebra of the corresponding conformal field theory. For models with $c<1$, a complete classification of critical exponents can be given in terms of the unitary series with central charge[2,6]

$$c=1-\frac{6}{h(h-1)} \tag{1.1}$$

where $h=4, 5, 6,\ldots$. In particular, the scaling dimensions of various scaling fields are determined by the conformal weights in the Kac table.[7]

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0022-4715/91/0700-0013$06.50/0

In lattice calculations, the central charge and scaling dimensions are accessible[8,9] from finite-size corrections to the row transfer matrix eigenvalues at criticality. Indeed, extensive calculations of these finite-size corrections[10–21] have given the strongest and most direct evidence to date supporting the predictions of conformal and modular invariance. However, apart from the Ising model (see, for example, ref. 22), there has been no exact analytic calculation of the scaling dimensions. The widely adopted methods for calculating the central charge based on the Euler–Maclaurin formula[10–13] or thermodynamics of the corresponding quantum chain[9,23] have proved too cumbersome to extend to calculations of the scaling dimensions.

In this paper, we develop methods to calculate analytically the finite-size corrections and scaling dimensions of critical lattice models as announced previously.[24] The results presented here bring to fruition an approach initiated in earlier work.[25] The methods are quite general and applications to other models will be given in subsequent papers. In this paper, as a first example, the methods are applied to the tricritical hard square and critical hard hexagon models. These models are both special cases of the generalized hard hexagon models introduced by Baxter[26–29] and further studied by Baxter and Pearce.[30,31] The results we obtain for the central charges and scaling dimensions are summarized as:

Tricritical hard squares: $c = 7/10, \quad x = 3/40, 1/5, 7/8, 6/5$ (1.2)

Critical hard hexagons: $c = 4/5, \quad x = 2/15, 4/5, 17/15, 4/3, 9/5$ (1.3)

The layout of the paper is as follows. The interacting hard square and hard hexagon lattice models are defined in Section 1.1. This section introduces the inversion identity satisfied by the row transfer matrices of these models and summarizes the known critical behavior. In Section 1.2 we discuss the predictions of conformal and modular invariance. The details of the calculations of the central charge and scaling dimensions of the two models are given separately in Sections 2 and 3. The paper concludes, in Section 4, with a discussion of further work to be done.

1.1. Generalized Hard Square and Hard Hexagon Models

The hard square and hard hexagon models are both special cases of a generalized interaction-round-a-face model[26,29–31] with face weights given by

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix} = z^{(a+b+c+d)/4} \exp(Lac + Mbd)(1-ab)(1-bc)(1-cd)(1-da) \tag{1.4}$$

This is a lattice gas on the square lattice with nearest-neighbor exclusion. Here z is the activity, L and M are diagonal interactions, and the spins or occupation numbers $a, b, c, d = 0, 1$ according to whether the site is empty or occupied. The usual noninteracting hard square model is given by $L = M = 0$ and the hard hexagon model is given by $L = 0$, $M = -\infty$.

The generalized hard square or hexagon model (1.4) satisfies a Yang–Baxter equation, and is therefore exactly solvable, when the interactions satisfy the constraint

$$z = (1 - e^{-L})(1 - e^{-M})/(e^{L+M} - e^{L} - e^{M}) \tag{1.5}$$

This surface consists of disjoint sheets with a line of critical points on each sheet given by

$$\Delta = z^{-1/2}(1 - ze^{L+M}) = \pm[(\sqrt{5} - 1)/2]^{5/2} \tag{1.6}$$

On the critical lines the interactions can be parametrized as

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix} = \frac{\sin(\lambda - u)}{\sin \lambda}\,\delta(a, c) + \frac{\sin u}{\sin \lambda}\left(\frac{S_a S_c}{S_b S_d}\right)^{1/2}\delta(b, d) \tag{1.7}$$

where

$$\lambda = \pi/5, \qquad S_1 = \sin \lambda, \qquad S_0 = \sin 2\lambda \tag{1.8}$$

The spectral parameter u is related to spatial anisotropy, λ is the crossing parameter, and δ is the Kronecker delta. If $L, M > 0$, then $0 \leqslant u \leqslant \pi/5$ and the model describes interacting hard squares. If $L > 0$, $M < 0$, then $-\pi/5 \leqslant u \leqslant 0$ and the model describes interacting hard hexagons. Setting $L = 0$, $M = -\infty$ in the critical condition (1.6) gives the critical activity of hard hexagons as

$$z_c = [\tfrac{1}{2}(1 + \sqrt{5})]^5 = \tfrac{1}{2}(11 + 5\sqrt{5}) = 11.09017\ldots \tag{1.9}$$

Similarly, taking $L = M > 0$ or $u = \pi/10$, we obtain a critical point of the isotropic interacting hard square model as

$$L_t = M_t = \ln(3 + \sqrt{5}) = 1.65557\ldots, \qquad z_t = (\sqrt{5} - 1)/32 = 0.038627\ldots \tag{1.10}$$

This is in fact a tricritical point.[31,32] The phase diagram of the isotropic hard square model with nearest-neighbor attractions is shown in Fig. 1.

The generalized hard square or hexagon model possesses a family of commuting row transfer matrices parametrized by the spectral parameter u. Suppose that σ and σ' are allowed spin configurations of two consecutive

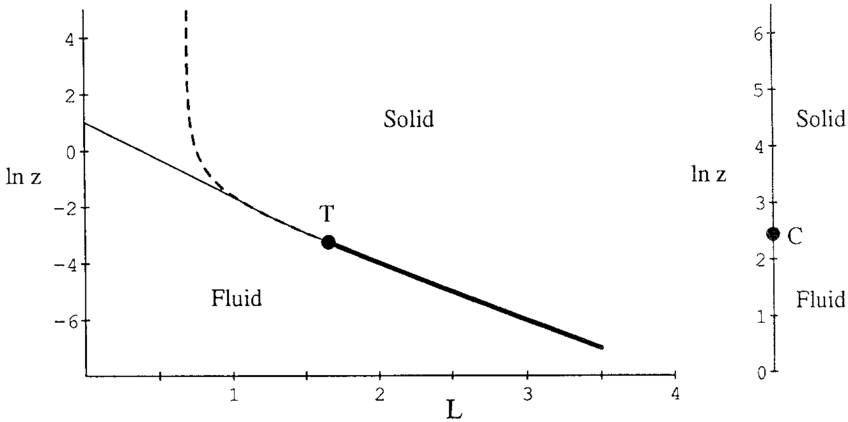

Fig. 1. Phase diagram of the isotropic interacting hard square model showing the phase boundary between the fluid and square ordered solid phases. The tricritical point T separates the critical line (solid curve) from the first-order line (thick solid curve). The model is exactly solvable on the first-order line and its analytic continuation (dashed line). The phase diagram of hard hexagons with its simple critical point C is shown on the right.

rows of an N-column lattice with periodic boundary conditions. Then the elements of the row transfer matrix are given by

$$\langle\sigma|\, \mathbf{V}(u)\, |\sigma'\rangle = \prod_{j=1}^{N} W\begin{pmatrix} \sigma_j' & \sigma_{j+1}' \\ \sigma_j & \sigma_{j+1} \end{pmatrix} \tag{1.11}$$

with $\sigma_{N+1} = \sigma_1$, $\sigma'_{N+1} = \sigma'_1$. The eigenvalues $V(u)$ of the transfer matrix $\mathbf{V}(u)$ are entire functions of u. They are completely determined by their zeros via the factorization

$$V(u) = R \prod_{j=1}^{N} \sin(u - u_j) \tag{1.12a}$$

where R is a constant independent of u. It can be shown that the zeros satisfy the sum rule

$$\sum_{j=1}^{N} u_j = 3N\lambda + k\pi, \qquad k \in \mathbb{Z}. \tag{1.12b}$$

Let us define a normalized transfer matrix $\mathbf{T}(u)$ by

$$\mathbf{T}(u) = \left[-\frac{\sin(2\lambda + u) \sin \lambda}{\sin(2\lambda - u) \sin(\lambda + u)} \right]^N \mathbf{V}(u) \tag{1.13}$$

Then these transfer matrices commute,

$$[\mathbf{T}(u), \mathbf{T}(v)] = 0 \tag{1.14}$$

and furthermore the eigenvalues $T(u)$ of $\mathbf{T}(u)$ satisfy[30,31] the special functional equation or inversion identity

$$T(u)\,T(u+\lambda)=1+T(u-2\lambda) \tag{1.15}$$

The eigenvalues $T(u)$ are in general complex. We will regard the eigenvalues $T(u)$ as functions of a complex variable u. They are then meromorphic functions of u satisfying the following periodicity and crossing symmetry:

$$T(u+\pi)=T(u), \qquad \bar{T}(u)=T(\lambda-\bar{u}) \tag{1.16}$$

If $u=u_z$ is a zero of $T(u)$, then it must satisfy the equations

$$T(u_z\pm 2\lambda)=-1, \qquad T(u_z\pm 3\lambda)=-1 \tag{1.17}$$

This is a consequence of the inversion identity and periodicity. The inversion identity (1.15) is an exact equation for finite N and completely determines the eigenvalues $T(u)$. It therefore contains all the information needed to calculate finite-size corrections. We will solve it using Fourier transforms. The eigenvalues $T(u)$ are analytic and nonzero in various vertical strips in the complex u plane. Moreover, taking the limit $\mathrm{Im}(u)\to\pm\infty$ in (1.15) and solving the resulting quadratic equation, we see that the asymptotic values are given by the golden numbers

$$\lim_{\mathrm{Im}(u)\to\infty} T(u)=\lim_{\mathrm{Im}(u)\to-\infty} T(u)=(1\pm\sqrt{5})/2 \tag{1.18}$$

It therefore follows that within such strips the derivative $[\ln T(u)]'$ admits a Fourier transform.

An inversion identity of the form (1.15) holds, not just at criticality, but on the whole of the exact solution manifold (1.5). Indeed, the inversion identity has already been solved off criticality[30,31] to obtain the free energy and correlation lengths of both interacting hard squares and hard hexagons. Similarly, the order parameters and sublattice densities of these models have been calculated using corner transfer matrices.[26,28,31] In summary, these results yield the following values for the critical exponents and scaling dimensions.

Tricritical hard squares:

$$\alpha=-1/2, \qquad \nu=5/4, \qquad \beta=3/32, \qquad \beta'=1/4$$
$$x_\varepsilon=6/5, \qquad x_\sigma=3/40, \qquad x'_\sigma=1/5 \tag{1.19}$$

Hard hexagons:

$$\alpha = 1/3, \qquad \nu = 5/6, \qquad \beta = 1/9$$
$$x_\varepsilon = 4/5, \qquad x_\sigma = 2/15 \tag{1.20}$$

In two dimensions, the thermal and magnetic scaling dimensions, denoted by x_ε and x_σ, respectively, are given in terms of the critical exponents by the scaling relations

$$2 - \alpha = 2\nu = \frac{2}{2 - x_\varepsilon}, \qquad \frac{2\beta}{2 - \alpha} = x_\sigma \tag{1.21}$$

1.2. Conformal and Modular Invariance

In this section we briefly summarize the application of conformal and modular invariance to critical lattice models with central charge $c < 1$. An emphasis is placed on the particular consequences of the theory for the row transfer matrix eigenvalue spectra of tricritical hard squares and critical hard hexagons. Comprehensive reviews are given elsewhere.(33–35)

Consider a critical lattice model with central charge $c < 1$. The central charge is therefore restricted to the values $c = 1 - 6/h(h-1)$. On a finite $M \times N$ periodic lattice or torus the partition function can be written as

$$Z_{M,N} = \exp(-MNf)\, Z(q) \tag{1.22}$$

where f is the bulk free energy and $Z(q)$ is a universal term describing the leading finite-size corrections in the limit of M, N large with the aspect ratio $\delta = M/N$ fixed. The argument q is the modular parameter. For a spatially isotropic model, it is simply related to the aspect ratio δ by $q = \exp(-2\pi\delta)$. The partition function on a torus, calculated from the eigenvalues of the row transfer matrix $\mathbf{T}$ of a periodic row of N faces, is

$$Z_{M,N} = \operatorname{Tr} \mathbf{T}^M = \sum_n \Lambda_n^M = \sum_n \exp(-ME_n) \tag{1.23}$$

where

$$\Lambda_n = \exp(-E_n) \tag{1.24}$$

are the eigenvalues of $\mathbf{T}$ and E_n are the corresponding energy levels labeled by $n = 0, 1, 2, \ldots$. The leading finite-size corrections to the energy levels take the form(8,9)

$$E_0 = Nf - \frac{\pi c}{6N} \sin\theta$$
$$E_n - E_0 = \frac{2\pi}{N} (x_n \sin\theta + i s_n \cos\theta) \tag{1.25}$$

where E_0 is the ground-state energy and c is the central charge. The angle θ is determined by the spatial anisotropy[36] and depends on the spectral parameter u. For tricritical hard squares and hard hexagons it is given by

$$\theta = \begin{cases} 5u, & 0 \leqslant u \leqslant \pi/5 \\ -10u/3, & -\pi/5 \leqslant u \leqslant 0 \end{cases} \tag{1.26}$$

The scaling dimensions and spins are

$$x_n = \Delta + \bar{\Delta} + k + \bar{k}, \qquad s_n = \Delta - \bar{\Delta} + k - \bar{k}, \qquad k, \bar{k} \in \mathbb{N} \tag{1.27}$$

where the conformal weights $(\Delta, \bar{\Delta})$ of the primary operators are given by the Kac formula[7]

$$\Delta = \Delta_{r,s} = \frac{[hr - (h-1)s]^2 - 1}{4h(h-1)}, \qquad 1 \leqslant r \leqslant h-2, \quad 1 \leqslant s \leqslant h-1, \quad r \geqslant s \tag{1.28}$$

with $\bar{\Delta}_{r,s} = \Delta_{\bar{r},\bar{s}}$ defined similarly. The allowed values of the conformal weights describing the critical behavior of the Ising model ($h = 4$), tricritical Ising model ($h = 5$), and tetracritical Ising model ($h = 6$) are listed in Table I. The tricritical hard square model is described by the $h = 5$ grid, has central charge $c = 7/10$, and lies in the universality class of the tricritical Ising model. The conformal weights of critical hard hexagons appear on

Table I. Grids of Conformal Weights $\Delta_{r,s}$ for $h = 4$, 5, and 6[a]

$h = 4$

	$r=1$	$r=2$
$s=2$		1/16
$s=1$	0	1/2

$h = 5$

	$r=1$	$r=2$	$r=3$
$s=3$			1/10
$s=2$		3/80	3/5
$s=1$	0	7/16	3/2

$h = 6$

	$r=1$	$r=2$	$r=3$	$r=4$
$s=4$				1/8
$s=3$			1/15	2/3
$s=2$		1/40	21/40	13/8
$s=1$	0	2/5	7/5	3

[a] The conformal weights of the tricritical hard square model are given by the $h = 5$ grid. The conformal weights of hard hexagons appear on odd rows of the $h = 6$ grid.

odd rows of the $h=6$ grid and so the hard hexagon model has central charge $c=4/5$. The hard hexagon model is in the universality class of the three-state Potts model.

The energy spectrum of $c<1$ critical lattice models consists of a ground state and an excited state for each primary operator. Above each of these is a tower of equally spaced levels or descendants. More specifically, the universal finite-size partition function is given as a sesquilinear form in Virasoro characters

$$Z(q)=\sum_{\Delta,\bar{\Delta}} \chi_{\Delta}(q)\,\mathcal{N}(\Delta,\bar{\Delta})\,\chi_{\bar{\Delta}}(\bar{q}) \tag{1.29}$$

where the sum is over conformal weights in the Kac table and the integer $\mathcal{N}(\Delta,\bar{\Delta})$ gives the multiplicity of the primary operator $(\Delta,\bar{\Delta})$. The modular parameter is

$$q=\exp(2\pi i\tau), \qquad \tau=\frac{M}{N}\exp[i(\pi-\theta)] \tag{1.30}$$

and $\bar{q}$ denotes the complex conjugate. The Virasoro characters are defined by

$$\begin{aligned}\chi_{\Delta}(q)=\chi_{\Delta_{r,s}}(q)&=q^{-c/24}\sum_{k=0}^{\infty} d_{\Delta}(k)q^{\Delta+k}\\ &=\frac{q^{-c/24}}{\prod_{n=1}^{\infty}(1-q^n)}\sum_{n=-\infty}^{\infty}\Big\{q^{\{[2h(h-1)n+hr-(h-1)s]^2-1\}/4h(h-1)}\\ &\quad -q^{\{[2h(h-1)n+hr+(h-1)s]^2-1\}/4h(h-1)}\Big\}\end{aligned} \tag{1.31}$$

The factors $d_{\Delta}(k)$ are integers giving the degeneracy of levels. The set of numbers $\mathcal{N}(\Delta,\bar{\Delta})\in\mathbb{N}$, which completely determines $Z(q)$, is called the operator content of the theory.

On a torus, the universal partition function (1.29) must be invariant under transformations of the modular group. This fact has been used to classify completely the modular invariant partition functions $Z(q)$ and operator content of theories with central charge $c<1$.[37–39] Remarkably, they are in a one-to-one correspondence with the classical *A-D-E* Lie algebras. There are two infinite families corresponding to the *A* and *D* series and three exceptional cases corresponding to the classical *E* Lie algebras. Critical lattice models corresponding to each of the *A-D-E* Lie algebras have been constructed by Pasquier.[40,41] The spin states of these models are described by the Dynkin diagrams of the associated Lie algebras. The A series is realized by the restricted solid-on-solid (RSOS) models of Andrews *et al.*[42] The Ising model ($h=4$) and tricritical hard square model ($h=5$) are the first two members of this hierarchy. At criticality, the face weights of the RSOS models are given by (1.7) with $a, b, c, d = 1, 2, \ldots, h-1$, $\lambda = \pi/h$, and $S_a = \sin(a\pi/h)$. The numbers S_a are the components of the Perron–Frobenius eigenvector of the adjacency matrix associated with the Dynkin diagram. The graphs of allowed neighboring states for interacting hard squares and hard hexagons are shown in Fig. 2.

The modular invariant partition functions of the classical A or RSOS models are

$$Z(q) = \sum_{\Delta} |\chi_\Delta(q)|^2 \tag{1.32}$$

In particular, the modular invariant partition function of tricritical hard squares is

$$\begin{aligned} Z(q) &= |\chi_0(q)|^2 + |\chi_{3/80}(q)|^2 + |\chi_{1/10}(q)|^2 + |\chi_{7/16}(q)|^2 + |\chi_{3/5}(q)|^2 + |\chi_{3/2}(q)|^2 \\ &= (q\bar{q})^{-7/240}[1 + (q\bar{q})^{3/80} + (q\bar{q})^{1/10} + (q\bar{q}\ )^{7/16} + (q^{3/80}\bar{q}^{83/80} + q^{83/80}\bar{q}^{3/80}) \\ &\quad + (q\bar{q})^{3/5} + (q^{1/10}\bar{q}^{11/10} + q^{11/10}\bar{q}^{1/10}) + (q^{7/16}\bar{q}^{23/16} + q^{23/16}\bar{q}^{7/16}) \\ &\quad + (q^2 + \bar{q}^2) + O(|q|^{83/40})] \end{aligned} \tag{1.33}$$

For the isotropic case with $u = \pi/10$, the modular parameter q is real and the first few terms simplify to

$$Z(q) = q^{-7/20}[1 + q^{3/40} + q^{1/5} + q^{7/8} + 2q^{43/40} + 3q^{6/5} + 2q^{15/8} + 2q^2 + O(q^{83/40})] \tag{1.34}$$

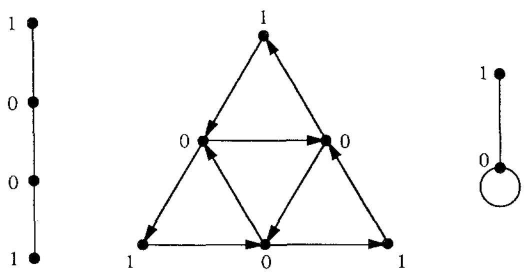

Fig. 2. Graphs of allowed neighboring states for interacting hard squares and hard hexagons. Note that the bonds in the hard hexagon case are directed. Identifying corresponding states under the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ symmetries leads to the tadpole diagram on the right for the two states 0 and 1 used in this paper.

On the other hand, the modular invariant partition function of the three-state Potts and hard hexagon models belongs to the D series and is given by

$$\begin{aligned} Z(q) &= |\chi_0(q) + \chi_3(q)|^2 + |\chi_{2/5}(q) + \chi_{7/5}(q)|^2 + 2\,|\chi_{1/15}(q)|^2 + 2\,|\chi_{2/3}(q)|^2 \\ &= (q\bar{q})^{-1/30}[1 + 2(q\bar{q})^{1/15} + (q\bar{q})^{2/5} + 2(q^{1/15}\bar{q}^{16/15} + q^{16/15}\bar{q}^{1/15}) \\ &\quad + 2(q\bar{q})^{2/3} + 2(q^{2/5}\bar{q}^{7/5} + q^{7/5}\bar{q}^{2/5}) + (q^2 + \bar{q}^2) + O(|q|^{32/15})] \end{aligned} \tag{1.35}$$

This completes the review of the application of the theory of conformal and modular invariance. To summarize, the modular invariant partition functions (1.33) and (1.35) are the predicted spectrum-generating functions of tricritical hard squares and critical hard hexagons. Most importantly, the scaling dimensions, spins, and degeneracies of energy levels associated to all the relevant scaling fields can be read off immediately from these expansions. In the following sections we will confirm these predictions by direct calculation of the finite-size corrections to row transfer matrix eigenvalues of tricritical hard squares and critical hard hexagons.

2. TRICRITICAL HARD SQUARES

In this section we describe our method for calculating finite-size corrections to the eigenvalues of the transfer matrix for tricritical hard squares. The physical regime is $0 \leqslant u \leqslant \pi/5$. We first consider the largest eigenvalue to calculate the central charge. The generalization to calculate the scaling dimensions of next-largest eigenvalues is presented in subsequent subsections. We will take N to be even throughout this section.

2.1. The Central Charge $c = 7/10$

The keystone in our approach is the identification of analyticity domains of the eigenvalues $T(u)$. These domains are delimited by curves in the complex u plane on which the zeros of the largest eigenvalue $T(u)$ become densely distributed in the thermodynamic limit. By periodicity we can restrict our attention to the strip $-\pi/10 \leqslant \mathrm{Re}(u) < 9\pi/10$. For tricritical hard squares, the zeros accumulate on the lines $\mathrm{Re}(u) = -\pi/10,\ 3\pi/10$ for large N as shown schematically in Fig. 3, where the distance of the furthest zeros from the real axis grows as $\ln N$. We therefore distinguish two analyticity strips:

$$\begin{aligned} &\text{strip 1:} \qquad -\frac{\pi}{10} < \mathrm{Re}(u) < \frac{3\pi}{10} \\ &\text{strip 2:} \qquad \frac{3\pi}{10} < \mathrm{Re}(u) < \frac{9\pi}{10} \end{aligned} \tag{2.1}$$

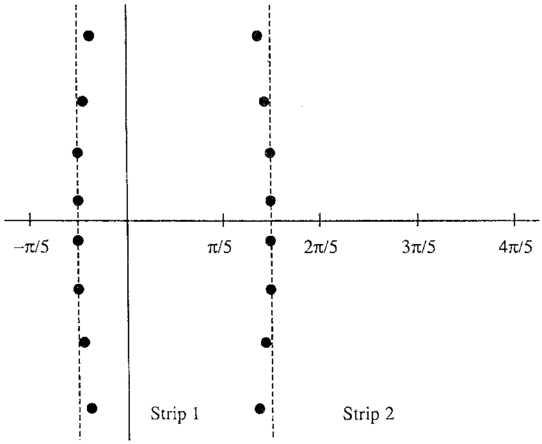

Fig. 3. Schematic representation of the zeros in the complex u plane of the largest eigenvalue of tricritical hard squares. The finite-size correction to this eigenvalue yields the central charge $c = 7/10$.

Since we are interested in finite-size corrections to the largest eigenvalue, it is natural to represent $T(u)$ by its bulk behavior multiplied by some correction function $l(u)$. Allowing for different analytic forms in strips 1 and 2 and using the known bulk behavior gives the representations

$$\begin{aligned} T_1(u) &= 1 \cdot l_1(u) \\ T_2(u) &= z(u)^N \cdot l_2(u), \qquad z(u) = \left[i \cot\left(\frac{5u}{2}\right)\right] \end{aligned} \tag{2.2}$$

Inserting (2.2) into the functional equation (1.15), we obtain relations for the l functions

$$\begin{aligned} l_1(u)\, l_1(u+\lambda) &= p_2(u) \\ l_2(u)\, l_2(u+\lambda) &= p_1(u) \end{aligned} \tag{2.3}$$

where the functions on the rhs are defined by

$$\begin{aligned} p_1(u) &:= 1 + T_1(u - 2\lambda) \\ p_2(u) &:= 1 + T_2(u + 3\lambda) \end{aligned} \tag{2.4}$$

As seen from (1.13), $l_1(u)$ and $l_2(u)$ are *a*nalytic, *n*onzero in strips 1 and 2, and their logarithms have *c*onstant asymptotic behavior for large positive or negative imaginary parts (ANZC). The functions p_1 and p_2 also share such an ANZC property in narrower strips. Therefore the derivatives of $\ln l_1$, etc., can be Fourier-transformed, e.g., with Fourier transform pair

$$\begin{aligned} L_1(k) &= \frac{1}{2\pi i}\int_{\mathrm{Re}(v)=x} dv\,[\ln l_1(v)]' e^{-kv} \\ [\ln l_1(v)]' &= \int_{-\infty}^{\infty} dk\, L_1(k) e^{kv} \end{aligned} \tag{2.5}$$

where the integration path denoted by $\mathrm{Re}(v)=x$ has to lie in the appropriate analyticity strip, but is otherwise arbitrary due to Cauchy's theorem. Taking the logarithm and derivative of (2.3) and then the Fourier transform, we obtain

$$\begin{aligned} L_1(k) + e^{\lambda k} L_1(k) &= P_2(k) \\ L_2(k) + e^{\lambda k} L_2(k) &= P_1(k) \end{aligned} \tag{2.6}$$

which is readily solved for the L functions in terms of the P functions. Transforming back, we find a double integral whose order can be interchanged. The k integral can then be evaluated using

$$\int_{-\infty}^{\infty} dk \frac{e^{ku}}{1+e^{k\pi/5}} = \frac{5}{\sin 5u}, \qquad 0 < \mathrm{Re}(u) < \frac{\pi}{5} \tag{2.7}$$

Thus we obtain

$$\begin{aligned} [\ln l_1(u)]' &= \frac{1}{2\pi i}\int_{\mathrm{Re}(v)=0} dv\,[\ln p_2(v)]' \frac{5}{\sin 5(u-v)} \\ [\ln l_2(u)]' &= \frac{1}{2\pi i}\int_{\mathrm{Re}(v)=\pi/2} dv\,[\ln p_1(v)]' \frac{5}{\sin 5(u-v)} \end{aligned} \tag{2.8}$$

These equations are valid in the strips $0 < \mathrm{Re}(u) < \pi/5$ and $\pi/2 < \mathrm{Re}(u) < 7\pi/10$ respectively.

To make further progress, it is convenient to restrict the T and p functions to certain straight lines in their analyticity strips in order to deal with functions of real variables

$$\begin{aligned} \mathfrak{a}(x) &:= T_1\left(\frac{i}{5}x + \frac{\pi}{10}\right), \qquad \mathfrak{A}(x) := p_1\left(\frac{i}{5}x + \frac{\pi}{2}\right) = 1 + \mathfrak{a}(x) \\ \mathfrak{b}(x) &:= T_2\left(\frac{i}{5}x + \frac{6\pi}{10}\right), \qquad \mathfrak{B}(x) := p_2\left(\frac{i}{5}x\right) = 1 + \mathfrak{b}(x) \end{aligned} \tag{2.9}$$

After integrating (2.8) with respect to u and introducing the functions defined in (2.9), we find

$$\begin{aligned} \ln \mathfrak{a}(x) &= k * \ln \mathfrak{B} + D_1 \\ \ln \mathfrak{b}(x) &= \ln z(ix/5 + 3\pi/5)^N + k * \ln \mathfrak{A} + D_2 \end{aligned} \tag{2.10}$$

where D_1 and D_2 are integration constants and the branch of the logarithm of z is chosen such that it vanishes as $x \to \infty$. The notation $g * f$ denotes the convolution of the functions g and f,

$$(g * f)(x) = \int_{-\infty}^{\infty} g(x-y)\, f(y)\, dy \tag{2.11}$$

and the kernel k is defined by

$$k(y) = \frac{1}{2\pi \cosh y} \tag{2.12}$$

To evaluate the constants D_1 and D_2 we take the limit $\mathrm{Im}(u) \to \pm\infty$ in (2.10) using the asymptotic values (1.18) of $T(u)$. Since the largest eigenvalue is positive, we must have

$$\mathfrak{a}(\infty) = \mathfrak{b}(\infty) = (1 + \sqrt{5})/2 \tag{2.13}$$

Using this and the integral

$$\int_{-\infty}^{\infty} dy\, k(y) = \frac{1}{2} \tag{2.14}$$

we deduce

$$D_1 = D_2 = 0 \tag{2.15}$$

To handle the thermodynamic limit we observe that z^N in (2.10) possesses the scaling behavior

$$\lim_{N \to \infty} z\left(\pm \frac{i}{5}(x + \ln N) + \frac{3\pi}{5} \right)^N = \exp(-2e^{-x}) \tag{2.16}$$

Numerically $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{A}$, and $\mathfrak{B}$ are found to scale similarly. We therefore define appropriate limiting functions in the positive scaling regime and introduce a shorthand notation for their logarithms:

$$\begin{aligned} a(x) &:= \lim_{N \to \infty} \mathfrak{a}(x + \ln N), & la(x) &:= \ln a(x) \\ A(x) &:= \lim_{N \to \infty} \mathfrak{A}(x + \ln N) = 1 + a(x), & lA(x) &:= \ln A(x) \\ b(x) &:= \lim_{N \to \infty} \mathfrak{b}(x + \ln N), & lb(x) &:= \ln b(x) \\ B(x) &:= \lim_{N \to \infty} \mathfrak{B}(x + \ln N) = 1 + b(x), & lB(x) &:= \ln B(x) \end{aligned} \tag{2.17}$$

Analogous limiting functions are defined in the negative scaling regime. In the case of the largest eigenvalue these are simply related, since $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{A}$, and $\mathfrak{B}$ are even functions. For the next-largest eigenvalues, however, such a relation is no longer valid in general and in later subsections we will treat the positive and negative scaling regimes separately.

In the positive scaling regime the integral equations (2.10) simplify to

$$\begin{aligned} la(x) &= k * lB \\ lb(x) &= -2e^{-x} + k * lA \end{aligned} \tag{2.18}$$

The notation could be made more compact by introducing a matrix K whose entries are functions

$$K := \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \tag{2.19}$$

We do not employ such a notation, but point out that the symmetry of K

$$K^T(y-x) = K(x-y) \tag{2.20}$$

will turn out to be a crucial property in further manipulations of (2.18).

Let us now consider the finite-size corrections to the eigenvalue in the physical strip $T_1(u)$. Using the variable $u = ix/5 + \pi/10$ in (2.9) and (2.10) and scaling the integration variable, we derive

$$\begin{aligned} \ln T_1(u) &= \ln \mathfrak{a}(x) = k * \ln \mathfrak{B}(x) \\ &= \int_0^\infty dy\, [k(x-y) + k(x+y)] \ln \mathfrak{B}(y) \\ &= \int_{-\ln N}^\infty dy\, [k(x-y-\ln N) + k(x+y+\ln N)] \ln \mathfrak{B}(y+\ln N) \\ &= \int_{-\infty}^\infty dy \left[\frac{e^{x-y}}{\pi N} + \frac{e^{-x-y}}{\pi N}\right] lB(y) + o\left(\frac{1}{N}\right) \\ &\simeq \frac{1}{\pi N}(e^x + e^{-x}) \int_{-\infty}^\infty dy\, e^{-y} lB(y) \end{aligned} \tag{2.21}$$

The finite-size corrections in the physical strip 1 are therefore determined by the behavior in the scaling regime of the unphysical strip 2. The integral in (2.21) is well-defined and nonzero. In particular, as $y \to -\infty$ the integrand vanishes

$$e^{-y} lB(y) = e^{-y} \ln[1 + \exp(-2e^{-y} + k * lA(y)] \to 0 \tag{2.22}$$

and is dominated by $\exp(-2e^{-y})$, so the integral exists. This confirms the scaling forms (2.17). A different ansatz in (2.17), for instance with $\ln N$ replaced by $\frac{1}{2}\ln N$ or $2\ln N$, would have led to correction terms of the form $0/\sqrt{N}$ or ∞/N^2, indicating an inconsistency.

The integral equations (2.18) can be solved straightforwardly numerically and appear to yield a unique solution. However, we have not been able to solve these equations analytically. Nonetheless, precisely the integral occurring in (2.21) can be calculated analytically from (2.18) without solving explicitly. We find this fortuitous fact remarkable. To achieve this, we first differentiate (2.18)

$$
\begin{aligned}
la'(x) &= k * lB' \\
lb'(x) &= 2e^{-x} + k * lA'
\end{aligned}
\tag{2.23}
$$

Multiplying (2.23) with $lA(x)$, $lB(x)$ and (2.18) with $lA'(x)$, $lB'(x)$, subtracting, and lastly integrating, we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty} dx\, [la'(x)\, lA(x) - la(x)\, lA'(x)] + \int_{-\infty}^{\infty} dx\, [lb'(x)\, lB(x) - lb(x)\, lB'(x)] \\
&\quad = 2\int_{-\infty}^{\infty} dx\, e^{-x}[lB(x) + lB'(x)]
\end{aligned}
\tag{2.24}
$$

where the contributions of the kernel K cancel due to the symmetry (2.20). After integrating by parts and using (2.22), the integral on the rhs of (2.24) is recognized as the required integral in (2.21). On the other hand, the integral on the lhs of (2.24) can be evaluated after changing the variable of integration x to a and b,

$$
\begin{aligned}
&4\int_{-\infty}^{\infty} dx\, e^{-x} lB(x) \\
&\quad = \int_{a(-\infty)}^{a(\infty)} da \left[\frac{\ln(1+a)}{a} - \frac{\ln a}{1+a}\right] + \int_{b(-\infty)}^{b(\infty)} db \left[\frac{\ln(1+b)}{b} - \frac{\ln b}{1+b}\right] \\
&\quad = 4L_+\left(\frac{1+\sqrt{5}}{2}\right) - 2L_+(1) = 4L\left(\frac{\sqrt{5}-1}{2}\right) - 2L\left(\frac{1}{2}\right) = \frac{7}{10}\frac{\pi^2}{3}
\end{aligned}
\tag{2.25}
$$

The last integrals are Rogers dilogarithms (Appendix A) and are evaluated by using the asymptotics of $a(x)$ and $b(x)$, which can be read off from (2.13) after recalling the definitions (2.17) and (2.18),

$$
\begin{aligned}
a(-\infty) &= 1, \qquad a(\infty) = (1+\sqrt{5})/2 \\
b(-\infty) &= 0, \qquad b(\infty) = (1+\sqrt{5})/2
\end{aligned}
\tag{2.26}
$$

Collecting together (2.21) and (2.25) and using the fact that

$$\cosh x = \sin 5u, \qquad \sinh x = i \cos 5u \tag{2.27}$$

gives the result

$$E_0 = -\ln T_1(u) \simeq -\frac{7}{10}\frac{\pi}{6N}\sin 5u \tag{2.28}$$

from which we see that

$$c = 7/10 \tag{2.29}$$

2.2. The Leading Magnetic Scaling Dimension $x = 3/40$

For tricritical hard squares the next-largest eigenvalues, or excitations from the ground state, possess $O(N/2)$ zeros on the lines $\mathrm{Re}(u) = -\pi/10$ and $3\pi/10$, respectively, and a finite number of zeros shifted onto the lines $\mathrm{Re}(u) = -\pi/5$, $\pi/10$, $2\pi/5$, $3\pi/5$ in the complex u plane. These patterns, which characterize the various eigenvalues, must satisfy the reflection symmetry about the axis $\mathrm{Re}(u) = \pi/10$. Clearly, there are many different possible patterns of zeros giving rise to the many possible excitations. As input for the calculation of these eigenvalues we will assume certain basic properties, such as the qualitative pattern of zeros in the complex u plane, as well as the asymptotic behavior of $T(u)$. These assumptions are readily justified for small values of N by direct numerical diagonalization of the transfer matrices.

The eigenvalue we calculate in this subsection is characterized by just two shifted zeros located exactly at the points $u = -\pi/5$ and $2\pi/5$ as shown in Fig. 4. In this case we are able to use almost the same analysis as for the largest eigenvalue. We work with two strips of analyticity labeled 1 and 2′, where the latter strip is just a little narrower than strip 2 in (2.1),

$$\begin{aligned} &\text{strip 1:} && -\frac{\pi}{10} < \mathrm{Re}(u) < \frac{3\pi}{10} \\ &\text{strip 2':} && \frac{2\pi}{5} < \mathrm{Re}(u) < \frac{4\pi}{5} \end{aligned} \tag{2.30}$$

We then derive the same integral equations (2.10); only the integration constants are different due to the different asymptotic behavior of $\mathfrak{a}$ and $\mathfrak{b}$,

$$\mathfrak{a}(\infty) = \mathfrak{b}(\infty) = (1 - \sqrt{5})/2 \tag{2.31}$$

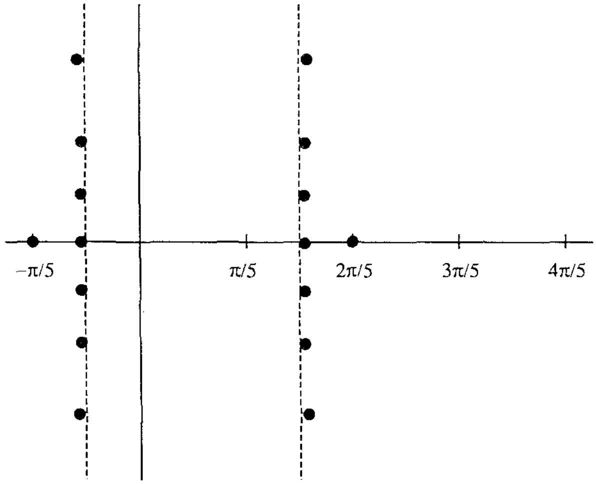

Fig. 4. Schematic representation of the zeros in the complex u plane of the leading magnetic eigenvalue of tricritical hard squares. The finite-size correction to this eigenvalue yields the scaling dimension $x = 3/40$.

We fix the branches of the functions $\ln \mathfrak{a}$ and $\ln \mathfrak{b}$ by setting

$$\ln \mathfrak{a}(\infty) = \ln \mathfrak{b}(\infty) = \ln[(\sqrt{5} - 1)/2] + \pi i \tag{2.32}$$

The logarithms of $\mathfrak{A}$ and $\mathfrak{B}$ are simpler to deal with, because $\mathfrak{A}$ and $\mathfrak{B}$ are real and positive for real arguments. The branches for $\ln \mathfrak{A}$ and $\ln \mathfrak{B}$ are fixed by

$$\begin{gathered} \ln \mathfrak{A}(\infty) = \ln \mathfrak{B}(\infty) = \ln[(3 - \sqrt{5})/2] \\ \ln \mathfrak{B}(0) = 0 \end{gathered} \tag{2.33}$$

where the last line is consistent with $\mathfrak{B}(0) = 1$.

From (2.32) and (2.33) we determine the constants in the integral equation (2.10),

$$D_1 = D_2 = \pi i \tag{2.34}$$

We define limiting functions in the positive scaling regime as in (2.17). The integral equations then reduce to

$$\begin{aligned} la(x) &= k * lB + \pi i \\ lb(x) &= -2e^{-x} + k * lA + \pi i \end{aligned} \tag{2.35}$$

We could have considered the negative scaling regime as well, but this is still not necessary, since the functions $\mathfrak{a}$ and $\mathfrak{b}$ are even. The finite-size correction to the eigenvalue in strip 1 is given in terms of lB by

$$\begin{aligned} \ln T_1(u) &= \ln \mathfrak{a}(x) = k * \ln \mathfrak{B}(x) + D_1 \\ &= \pi i + \int_0^\infty dy\, [k(x-y)+k(x+y)] \ln \mathfrak{B}(y) \\ &\simeq \pi i + \frac{1}{\pi N}(e^x + e^{-x}) \int_{-\infty}^{\infty} dy\, e^{-y}\, lB(y) \end{aligned} \tag{2.36}$$

Multiplying the derivative of (2.35) with $lA(x)$, $lB(x)$ and (2.35) with $lA'(x)$, $lB'(x)$, subtracting, and lastly integrating, we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} dx\, [la'(x)\, lA(x) - \{la(x) - \pi i\}\, lA'(x)] \\ &\quad + \int_{-\infty}^{\infty} dy\, [lb'(x)\, lB(x) - \{lb(x) - \pi i\}\, lB'(x)] \\ &= 2 \int_{-\infty}^{\infty} dx\, e^{-x} [lB(x) + lB'(x)] \end{aligned} \tag{2.37}$$

After performing an integration by parts and changing the variable of integration x to a and b, we obtain

$$\begin{aligned} &4 \int_{-\infty}^{\infty} dx\, e^{-x} lB(x) \\ &= \int_{a(-\infty)}^{a(\infty)} da \left[\frac{\ln(1+a)}{a} - \frac{\ln(-a)}{1+a}\right] + \int_{b(-\infty)}^{b(\infty)} db \left[\frac{\ln(1+b)}{b} - \frac{\ln(-b)}{1+b}\right] \\ &= \int_{-a(-\infty)}^{-a(\infty)} da \left[\frac{\ln(1-a)}{a} + \frac{\ln a}{1-a}\right] + \int_{-b(-\infty)}^{-b(\infty)} db \left[\frac{\ln(1-b)}{b} + \frac{\ln b}{1-b}\right] \\ &= -4L\left(\frac{\sqrt{5}-1}{2}\right) + 2L(1) = -\frac{1}{5}\frac{\pi^2}{3} \end{aligned} \tag{2.38}$$

The dilogarithm integrals are calculated using the asymptotics of $a(x)$ and $b(x)$,

$$\begin{aligned} a(-\infty) &= -1, \qquad a(\infty) = (1-\sqrt{5})/2 \\ b(-\infty) &= 0, \qquad b(\infty) = (1-\sqrt{5})/2 \end{aligned} \tag{2.39}$$

Combining (2.36) and (2.38) gives the result

$$E = -\ln T_1(u) \simeq -\pi i + \frac{1}{5}\frac{\pi}{6N}\sin 5u \tag{2.40}$$

or

$$E - E_0 = -\pi i + \frac{3}{40}\frac{2\pi}{N}\sin 5u \tag{2.41}$$

Recalling (1.25), we see that

$$x_1 = 3/40 \tag{2.42}$$

with spin zero.

2.3. The Leading Thermal Scaling Dimension $x = 1/5$

The eigenvalues $T(u)$ calculated in this subsection are characterized by two shifted zeros $u_\pm$ located exactly on the line $\text{Re}(u) = \pi/10$ as shown in Fig. 5. We are particularly interested in next-leading eigenvalues of this type and anticipate that they are given by zeros $u_\pm$ with imaginary parts

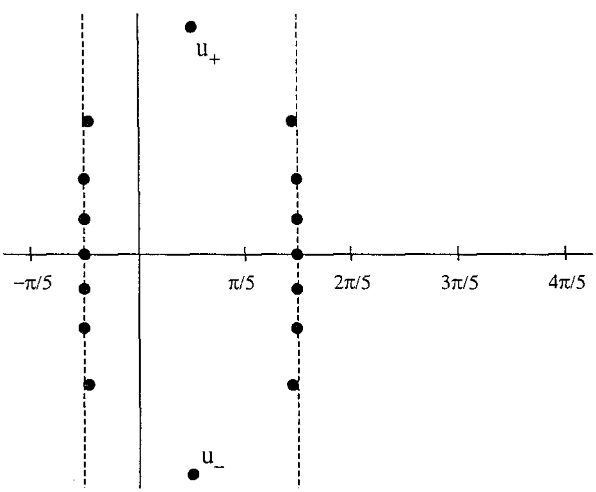

Fig. 5. Schematic representation of the zeros in the complex u plane of the leading thermal eigenvalue of tricritical hard squares. The finite-size correction to this eigenvalue yields the scaling dimension $x = 1/5$.

as large as possible. The location of the zeros $u_\pm$ is determined by the equation $T(u_\pm + 3\lambda) = -1$ as in (1.17). From this we see that $u_\pm$ have to scale with $\ln N$,

$$
\begin{aligned}
u_+ &= \frac{i}{5}(y_+ + \ln N) + \frac{\pi}{10} \\
u_- &= -\frac{i}{5}(y_- + \ln N) + \frac{\pi}{10}
\end{aligned} \tag{2.43}
$$

We represent $T(u)$ by explicitly taking into account the zeros $u_\pm$,

$$
\begin{aligned}
T_1(u) &= \tan \tfrac{5}{2}(u - u_+) \tan \tfrac{5}{2}(u - u_-) \cdot l_1(u) \\
T_2(u) &= z(u)^N \, l_2(u)
\end{aligned} \tag{2.44}
$$

where l_1 and l_2 are ANZC in strips 1 and 2 as in (2.1). From the functional equation (1.15) we get exactly the same relations for the l functions as in (2.3) and (2.4). We therefore take over the analysis of Section 2.1 through to Eq. (2.9). Instead of (2.10), we now have

$$
\begin{aligned}
\ln \mathfrak{a}(x) &= \ln[-\tanh \tfrac{1}{2}(x - y_+ - i\varepsilon - \ln N) \tanh \tfrac{1}{2}(x + y_- - i\varepsilon + \ln N)] \\
&\quad + k * \ln \mathfrak{B} + D_1 \\
\ln \mathfrak{b}(x) &= \ln z(ix/5 + 3\pi/5)^N + k * \ln \mathfrak{A} + D_2
\end{aligned} \tag{2.45}
$$

where the infinitesimally small $\varepsilon > 0$ is introduced so that the integration line avoids the two zeros $\pm(y_\pm + \ln N)$. The branch of the first logarithm on the rhs is fixed such that it approaches $\pm\pi i$ as $x \to \pm\infty$. We next evaluate the constants D_1, D_2 from the asymptotic behavior of $\mathfrak{a}$ and $\mathfrak{b}$,

$$
\mathfrak{a}(\pm\infty) = \mathfrak{b}(\pm\infty) = (1 - \sqrt{5})/2 \tag{2.46}
$$

The branches of the logarithms are fixed by requiring

$$
\ln \mathfrak{a}(\pm\infty) = \ln \mathfrak{b}(\pm\infty) = \ln[(\sqrt{5} - 1)/2] + \pi i \tag{2.47}
$$

and

$$
\begin{aligned}
\ln \mathfrak{A}(\pm\infty) = \ln \mathfrak{B}(\pm\infty) &= \ln[(3 - \sqrt{5})/2] \\
\ln \mathfrak{B}(0) &= 0
\end{aligned} \tag{2.48}
$$

where the last line is consistent with $\mathfrak{B}(0) = 1$. Using (2.47) and (2.48) in (2.45), we get

$$
D_1 = 0, \qquad D_2 = \pi i \tag{2.49}
$$

We next define limiting functions of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{A}$, and $\mathfrak{B}$ in the positive and negative scaling regimes by

$$\begin{aligned}
a_\pm(x) &:= \lim_{N\to\infty} \mathfrak{a}(\pm x \pm \ln N), & la_\pm(x) &:= \ln a_\pm(x)\\
A_\pm(x) &:= \lim_{N\to\infty} \mathfrak{A}(\pm x \pm \ln N) = 1 + a_\pm(x), & lA_\pm(x) &:= \ln A_\pm(x)\\
b_\pm(x) &:= \lim_{N\to\infty} \mathfrak{B}(\pm x \pm \ln N), & lb_\pm(x) &:= \ln b_\pm(x)\\
B_\pm(x) &:= \lim_{N\to\infty} \mathfrak{B}(\pm x \pm \ln N) = 1 + b_\pm(x), & lB_\pm(x) &:= \ln B_\pm(x)
\end{aligned} \tag{2.50}$$

where the branches are fixed such that

$$\begin{aligned}
la_\pm(\infty) = lb_\pm(\infty) &= \ln[(\sqrt{5}-1)/2] \pm \pi i\\
lA_\pm(\infty) = lB_\pm(\infty) &= \ln[(3-\sqrt{5})/2]\\
lB_\pm(-\infty) &= 0
\end{aligned} \tag{2.51}$$

From (2.45) it then follows immediately that

$$\begin{aligned}
la_\pm(x) &= \ln[-\tanh \tfrac{1}{2}(x - y_\pm \mp i\varepsilon)] + k * lB_\pm\\
lb_\pm(x) &= -2e^{-x} + k * lA_\pm \pm \pi i
\end{aligned} \tag{2.52}$$

Before proceeding we find the finite-size correction to the eigenvalue in the physical strip. The product of tan functions in (2.44) contributes a $1/N$ term for $\ln T_1$ of the form

$$\begin{aligned}
&\ln\left[-\tanh\frac{1}{2}(x - y_+ - i\varepsilon - \ln N)\tanh\frac{1}{2}(x + y_- - i\varepsilon + \ln N)\right]\\
&\quad \simeq -\frac{2}{N}(e^{x-y_+} + e^{-x-y_-})
\end{aligned} \tag{2.53}$$

Another $O(1/N)$ correction terms is given by

$$\begin{aligned}
k * \ln \mathfrak{B}(x) &= \int_0^\infty dy\, [k(x-y)\ln \mathfrak{B}(y) + k(x+y)\ln \mathfrak{B}(-y)]\\
&\simeq \frac{e^x}{\pi N}\int_{-\infty}^\infty dy\, e^{-y} lB_+(y) + \frac{e^{-x}}{\pi N}\int_{-\infty}^\infty dy\, e^{-y} lB_-(y)
\end{aligned} \tag{2.54}$$

Combining (2.53) and (2.54), we find

$$\ln T_1(u) \simeq \frac{1}{N}\left\{e^x\left[-2e^{-y_+}+\frac{1}{\pi}\int_{-\infty}^{\infty} dy\, e^{-y} lB_+(y)\right]\right.$$
$$\left.+e^{-x}\left[-2e^{-y_-}+\frac{1}{\pi}\int_{-\infty}^{\infty} dy\, e^{-y} lB_-(y)\right]\right\} \qquad (2.55)$$

To apply our manipulation of the integral equations, we need to express the exponentials in (2.55) in terms of integrals involving the functions $lA_\pm$. This is achieved by remembering the condition $T(u_z+3\pi/5)=-1$ given by (1.17) that each zero of $T(u)$ has to satisfy. Applying this requirement to $u_\pm$ in the scaling limit, we find the equivalent conditions

$$b_\pm\left(y_\pm \mp i\frac{\pi}{2}\right)=-1, \qquad \pm i\, lb_\pm\left(y_\pm \mp i\frac{\pi}{2}\right)=(2k_\pm-1)\pi \qquad (2.56)$$

where $k_\pm$ are integers. The lhs of the last equation can be written in terms of the functions $lA_\pm$ using (2.52)

$$\pm i\, lb_\pm\left(y_\pm \mp i\frac{\pi}{2}\right)=2e^{-y_\pm}\pm\frac{i}{2\pi}\int_{-\infty}^{\infty} dy\,\frac{lA_\pm(y)}{\cosh(y_\pm \mp i\pi/2-y)}-\pi \qquad (2.57)$$

Combining the last two equations, we see that

$$2e^{-y_\pm}=-\frac{1}{2\pi}\int_{-\infty}^{\infty} dy\,\frac{lA_\pm(y)}{\sinh(y-y_\pm)}+2k_\pm\pi \qquad (2.58)$$

Inserting this into (2.55) now gives

$$\ln T_1(u)$$
$$\simeq \frac{1}{N}\left\{e^x\left[-2k_+\pi+\frac{1}{2\pi}\int_{-\infty}^{\infty} dy\,\frac{lA_+(y)}{\sinh(y-y_+)}+\frac{1}{\pi}\int_{-\infty}^{\infty} dy\, e^{-y} lB_+(y)\right]\right.$$
$$\left.+e^{-x}\left[-2k_-\pi+\frac{1}{2\pi}\int_{-\infty}^{\infty} dy\,\frac{lA_-(y)}{\sinh(y-y_-)}+\frac{1}{\pi}\int_{-\infty}^{\infty} dy\, e^{-y} lB_-(y)\right]\right\}$$
$$(2.59)$$

The integrals appearing in (2.59) can in fact be calculated from (2.52) without solving (2.52) explicitly. After multiplying the derivative of (2.52) with $lA_\pm$, $lB_\pm$ and (2.52) with $lA'_\pm$, $lB'_\pm$, taking the difference, and integrating, we obtain

$$\int_{-\infty}^{\infty} dx\,[la'_{\pm}(x)\,lA_{\pm}(x) - la_{\pm}(x)\,lA'_{\pm}(x)]$$

$$+\int_{-\infty}^{\infty} dx\,\{lb'_{\pm}(x)\,lB_{\pm}(x) - [lb_{\pm}(x) \mp \pi i]\,lB'_{\pm}(x)\}$$

$$= \int_{-\infty}^{\infty} dx\,\{\ln[-\tanh \tfrac{1}{2}(x-y_{\pm})]'\,lA_{\pm}(x) - \ln[-\tanh \tfrac{1}{2}(x-y_{\pm})]\,lA'_{\pm}(x)\}$$

$$+2\int_{-\infty}^{\infty} dx\,e^{-x}[lB_{\pm}(x) + lB'_{\pm}(x)] \tag{2.60}$$

Performing an integration by parts using

$$\ln\left[\tanh\frac{1}{2}(x-y_{\pm})\right]' = \frac{1}{\sinh(x-y_{\pm})} \tag{2.61}$$

and $\ln[-\tanh(-\infty)] = 0$ as well as $\ln[-\tanh(\infty)] = \pm\pi i$, we then obtain

$$2\int_{-\infty}^{\infty} dx\,\frac{lA_{\pm}(x)}{\sinh(x-y_{\pm})} + 4\int_{-\infty}^{\infty} dx\,e^{-x} lB_{\pm}(x)$$

$$= \int_{-\infty}^{\infty} dx\,[la'_{\pm}(x)\,lA_{\pm}(x) - la_{\pm}(x)\,lA'_{\pm}(x)] \pm \pi i\,lA_{\pm}(\infty)$$

$$+\int_{-\infty}^{\infty} dx\,\{lb'_{\pm}(x)\,lB_{\pm}(x) - [lb_{\pm}(x) \mp \pi i]\,lB'_{\pm}(x)\} \tag{2.62}$$

Changing the variable of integration x to a and b on the right side of (2.62) gives the a integral

$$\int_{-\infty}^{\infty} dx\,[la'_{\pm}(x)\,lA_{\pm}(x) - la_{\pm}(x)\,lA'_{\pm}(x)] \pm \pi i\,lA_{\pm}(\infty)$$

$$= \int_{a(-\infty)}^{a(\infty)} da\left[\frac{\ln(1+a)}{a} - \frac{\ln a}{1+a}\right] \pm \pi i\,lA_{\pm}(\infty)$$

$$= \int_{0}^{(1-\sqrt{5})/2} da\left[\frac{\ln(1+a)}{a} - \frac{\ln(-a)}{1+a}\right] + \int_{1}^{0} da\left[\frac{\ln(1+a)}{a} - \frac{\ln a}{1+a}\right] \tag{2.63}$$

For the last step in (2.63) we have specified the correct branches of the logarithm and have used the asymptotics after dropping subscripts $\pm$,

$$\begin{aligned} a(-\infty) &= 1, \qquad & a(\infty) &= (1-\sqrt{5})/2 \\ b(-\infty) &= 0, \qquad & b(\infty) &= (1-\sqrt{5})/2 \end{aligned} \tag{2.64}$$

A similar calculation for the b integral in (2.62) yields

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\, &\{lb'_{\pm}(x)\, lB_{\pm}(x) - [lb_{\pm}(x) \mp \pi i]\, lB'_{\pm}(x)\} \\
&= \int_{b(-\infty)}^{b(\infty)} db \left[\frac{\ln(1+b)}{b} - \frac{\ln b \mp \pi i}{1+b}\right] \\
&= \int_{0}^{(1-\sqrt{5})/2} db \left[\frac{\ln(1+b)}{b} - \frac{\ln(-b)}{1+b}\right]
\end{aligned} \tag{2.65}$$

Inserting (2.63), (2.65) into (2.62), we derive

$$\begin{aligned}
2\int_{-\infty}^{\infty} &dx\, \frac{lA_{\pm}(x)}{\sinh(x-y_{\pm})} + 4\int_{-\infty}^{\infty} dx\, e^{-x} lB_{\pm}(x) \\
&= 2\int_{0}^{(\sqrt{5}-1)/2} da \left[\frac{\ln(1-a)}{a} + \frac{\ln a}{1-a}\right] - \int_{0}^{1} da \left[\frac{\ln(1+a)}{a} - \frac{\ln a}{1+a}\right] \\
&= -4L\left(\frac{\sqrt{5}-1}{2}\right) - 2L\left(\frac{1}{2}\right) = -\frac{17}{10}\frac{\pi^2}{3}
\end{aligned} \tag{2.66}$$

Recalling (2.59), this yields

$$E = -\ln T_1(u) \simeq \frac{2\pi}{N}\left[\left(k_+ + k_- + \frac{17}{120}\right)\sin 5u + (k_+ - k_-)i\cos 5u\right] \tag{2.67}$$

and

$$E - E_0 \simeq \frac{2\pi}{N}\left[\left(k_+ + k_- + \frac{1}{5}\right)\sin 5u + (k_+ - k_-)i\cos 5u\right] \tag{2.68}$$

Varying the integers k_+, k_- gives rise to a whole tower of eigenvalues of the transfer matrix with scaling dimensions $x = 1/5 + k_+ + k_-$ and spins $s = k_+ - k_-$ subject only to the physical constraint $x > 0$. In particular, the smallest choice $k_+ = k_- = 0$ gives the eigenvalue corresponding to the leading thermal scaling dimension

$$x_2 = 1/5 \tag{2.69}$$

with zero spin.

2.4. The Magnetic Scaling Dimension $x = 7/8$

In this subsection we calculate eigenvalues $T(u)$ characterized by four shifted zeros. Two of them are located at the points $u = -\pi/5,\ 2\pi/5$. The other two zeros $u_{\pm}$ lie on the line $\mathrm{Re}(u) = \pi/10$,

$$\begin{aligned} u_+ &= \frac{i}{5}(y_+ + \ln N) + \frac{\pi}{10} \\ u_- &= -\frac{i}{5}(y_- + \ln N) + \frac{\pi}{10} \end{aligned} \tag{2.70}$$

The pattern of zeros is depicted in Fig. 6. We respesent $T(u)$ exactly as in (2.44) where l_1 and l_2 are ANZC in strips 1 and 2′ given by (2.30). Proceeding as in the last subsection, we again derive the integral equations (2.45); however, the constants D_1, D_2 are different due to the different asymptotic behavior of $\mathfrak{a}$ and $\mathfrak{b}$,

$$\mathfrak{a}(\pm\infty) = \mathfrak{b}(\pm\infty) = (1+\sqrt{5})/2 \tag{2.71}$$

We fix the branches of the logarithms by requiring them to have real asymptotics. Using this in (2.45), we get

$$D_1 = -\pi i, \qquad D_2 = 0 \tag{2.72}$$

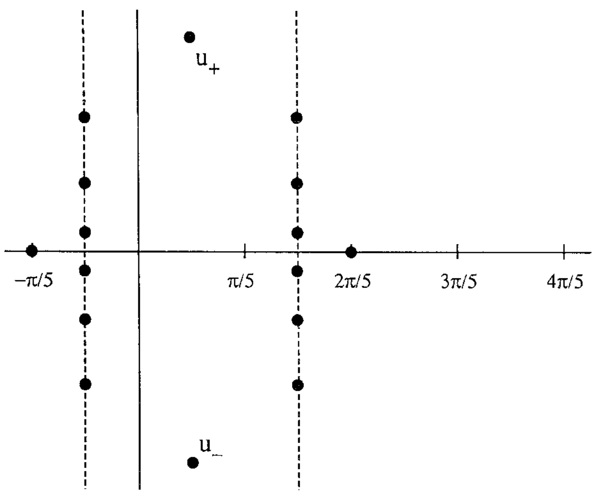

Fig. 6. Schematic representation of the zeros in the complex u plane of the second magnetic eigenvalue of tricritical hard squares. The finite-size correction to this eigenvalue yields the scaling dimension $x = 7/8$.

We see immediately that the limiting functions of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{A}$, and $\mathfrak{B}$ defined in (2.50) satisfy the integral equations

$$\begin{aligned} la_\pm(x) &= \ln[-\tanh \tfrac{1}{2}(x - y_\pm \mp i\varepsilon)] + k * lB_\pm \mp \pi i \\ lb_\pm(x) &= -2e^{-x} + k * lA_\pm \end{aligned} \tag{2.73}$$

where the branches are fixed such that

$$\begin{aligned} la_\pm(\infty) = lb_\pm(\infty) &= \ln[(1+\sqrt{5})/2] \\ lA_\pm(\infty) = lB_\pm(\infty) &= \ln[(3+\sqrt{5})/2] \\ lB_\pm(-\infty) &= 0 \end{aligned} \tag{2.74}$$

We can now carry through the previous analysis, noting that the term $-\pi$ in (2.57) is missing. Thus we derive the finite-size correction to the eigenvalue in strip 1,

$$\begin{aligned} \ln T_1(u) \simeq -\pi i + \frac{1}{N}\bigg\{ & e^x \bigg[-(2k_+ - 1)\pi + \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \, \frac{lA_+(y)}{\sinh(y - y_+)} \\ & + \frac{1}{\pi}\int_{-\infty}^{\infty} dy \, e^{-y} lB_+(y) \bigg] \\ & + e^{-x} \bigg[-(2k_- - 1)\pi + \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \, \frac{lA_-(y)}{\sinh(y - y_-)} \\ & + \frac{1}{\pi}\int_{-\infty}^{\infty} dy \, e^{-y} lB_-(y) \bigg] \bigg\} \end{aligned} \tag{2.75}$$

We next calculate the integrals in (2.75). We first multiply the derivative of (2.73) with $lA_\pm$, $lB_\pm$ and (2.73) with $lA'_\pm$, $lB'_\pm$. After taking the integral of the difference and performing an integration by parts, we obtain

$$\begin{aligned} & 2\int_{-\infty}^{\infty} dx \, \frac{lA_\pm(x)}{\sinh(x - y_\pm)} + 4\int_{-\infty}^{\infty} dx \, e^{-x} lB_\pm(x) \\ & = \int_{-\infty}^{\infty} dx \, \{ la'_\pm(x) \, lA_\pm(x) - [la_\pm(x) \pm \pi i] \, lA'_\pm(x) \} \pm \pi i \, lA_\pm(\infty) \\ & \quad + \int_{-\infty}^{\infty} dx \, [lb'_\pm(x) \, lB_\pm(x) - lb_\pm(x) \, lB'_\pm(x)] \end{aligned} \tag{2.76}$$

Changing the variable of integration x to a and b on the right side of (2.76), we get for the a integral

$$\int_{-\infty}^{\infty} dx \,\{la'_{\pm}(x)\, lA_{\pm}(x) - [la_{\pm}(x) \pm \pi i]\, lA'_{\pm}(x)\} \pm \pi i\, lA_{\pm}(\infty)$$

$$= \int_{a(-\infty)}^{a(\infty)} da \left[\frac{\ln(1+a)}{a} - \frac{\ln a \pm \pi i}{1+a}\right] \pm \pi i\, lA_{\pm}(\infty)$$

$$= \int_{0}^{(1+\sqrt{5})/2} da \left[\frac{\ln(1+a)}{a} - \frac{\ln a}{1+a}\right] + \int_{-1}^{0} da \left[\frac{\ln(1+a)}{a} - \frac{\ln(-a)}{1+a}\right] \tag{2.77}$$

where we have dropped the subscripts $\pm$ and used the asymptotics

$$\begin{aligned} a(-\infty) &= -1, \qquad & a(\infty) &= (1+\sqrt{5})/2 \\ b(-\infty) &= 0, \qquad & b(\infty) &= (1+\sqrt{5})/2 \end{aligned} \tag{2.78}$$

A similar calculation for the b integral in (2.76) yields

$$\int_{-\infty}^{\infty} dx\, [lb'_{\pm}(x)\, lB_{\pm}(x) - lb_{\pm}(x)\, lB'_{\pm}(x)]$$

$$= \int_{b(-\infty)}^{b(\infty)} db \left[\frac{\ln(1+b)}{b} - \frac{\ln b}{1+b}\right]$$

$$= \int_{0}^{(1+\sqrt{5})/2} db \left[\frac{\ln(1+b)}{b} - \frac{\ln b}{1+b}\right] \tag{2.79}$$

Inserting (2.77), (2.79) into (2.76), we derive

$$2\int_{-\infty}^{\infty} dx \frac{lA_{\pm}(x)}{\sinh(x-y_{\pm})} + 4\int_{-\infty}^{\infty} dx\, e^{-x} lB_{\pm}(x)$$

$$= 2\int_{0}^{(1+\sqrt{5})/2} da \left[\frac{\ln(1+a)}{a} - \frac{\ln a}{1+a}\right] - \int_{0}^{1} da \left[\frac{\ln(1-a)}{a} + \frac{\ln a}{1-a}\right]$$

$$= 4L\left(\frac{\sqrt{5}-1}{2}\right) + 2L(1) = \frac{11}{5}\frac{\pi^2}{3} \tag{2.80}$$

Recalling (2.75), this yields

$$E = -\ln T_1(u) \simeq \pi i + \frac{2\pi}{N}\left[\left(k_+ + k_- - \frac{71}{60}\right)\sin 5u + (k_+ - k_-)\, i\cos 5u\right] \tag{2.81}$$

and

$$E-E_0 \simeq \frac{2\pi}{N}\left[\left(k_+ + k_- - \frac{9}{8}\right)\sin 5u + (k_+ - k_-)i\cos 5u\right] \tag{2.82}$$

Again varying the integers k_+, k_- gives rise to a whole tower of eigenvalues of the transfer matrix. The choice giving the smallest positive scaling dimension is $k_+ = k_- = 1$. This leads to the magnetic scaling dimension

$$x_3 = 7/8 \tag{2.83}$$

with zero spin.

2.5. The Thermal Scaling Dimension x = 6/5

The eigenvalues $T(u)$ we calculate in this subsection are also characterized by four shifted zeros. Two of these, $u_{1\pm}$, are located on the line $\mathrm{Re}(u)=\pi/10$ and the other two zeros, $u_{2\pm}$, are located on the line $\mathrm{Re}(u)=3\pi/5$,

$$\begin{aligned} u_{1+} &= \frac{i}{5}(y_{1+}+\ln N)+\frac{\pi}{10}, & u_{2+} &= \frac{i}{5}(y_{2+}+\ln N)+\frac{3\pi}{5} \\ u_{1-} &= -\frac{i}{5}(y_{1-}+\ln N)+\frac{\pi}{10}, & u_{2-} &= -\frac{i}{5}(y_{2-}+\ln N)+\frac{3\pi}{5} \end{aligned} \tag{2.84}$$

The pattern of zeros is shown in Fig. 7. We represent $T(u)$ by explicitly taking into account the zeros $u_{1\pm}$ and $u_{2\pm}$,

$$\begin{aligned} T_1(u) &= \tan \tfrac{5}{2}(u-u_{1+})\tan \tfrac{5}{2}(u-u_{1-})\cdot l_1(u) \\ T_2(u) &= z(u)^N \tan \tfrac{5}{2}(u-u_{2+})\tan \tfrac{5}{2}(u-u_{2-})\cdot l_2(u) \end{aligned} \tag{2.85}$$

Here l_1 and l_2 are ANZC in strips 1 and 2′ as given in (2.30). From the functional equation (1.15) we obtain exactly the same relations for the l functions as in (2.3) and (2.4). The analysis of Section 2.1 therefore holds through to (2.9). Instead of (2.10) we now obtain

$$\begin{aligned} \ln \mathfrak{a}(x) &= \ln[-\tanh \tfrac{1}{2}(x-y_{1+}-i\varepsilon-\ln N)\tanh \tfrac{1}{2}(x+y_{1-}-i\varepsilon+\ln N)] \\ &\quad + k * \ln \mathfrak{B} + D_1 \\ \ln \mathfrak{b}(x) &= \ln z(ix/5+3\pi/5)^N \\ &\quad + \ln[-\tanh \tfrac{1}{2}(x-y_{2+}-i\varepsilon-\ln N)\tanh \tfrac{1}{2}(x+y_{2-}-i\varepsilon+\ln N)] \\ &\quad + k * \ln \mathfrak{A} + D_2 \end{aligned} \tag{2.86}$$

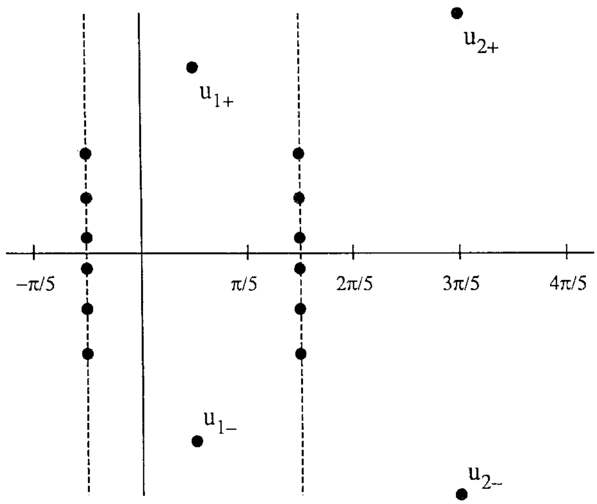

Fig. 7. Schematic representation of the zeros in the complex u plane of the second thermal eigenvalue of tricritical hard squares. The finite-size correction to this eigenvalue yields the scaling dimension $x = 6/5$.

where the infinitesimally small $\varepsilon > 0$ is introduced so that the integration line avoids the zeros $\pm(y_{1,2\pm} + \ln N)$. The constants D_1, D_2 can be evaluated from the asymptotic behavior of $\mathfrak{a}$ and $\mathfrak{b}$,

$$\mathfrak{a}(\pm\infty) = \mathfrak{b}(\pm\infty) = (1 - \sqrt{5})/2 \tag{2.87}$$

The branches of logarithms are fixed by

$$\ln \mathfrak{a}(\infty) = \ln \mathfrak{b}(\infty) = \ln[(\sqrt{5} - 1)/2] + \pi i \tag{2.88}$$

and

$$\begin{gathered} \ln \mathfrak{A}(\pm\infty) = \ln \mathfrak{B}(\pm\infty) = \ln[(3 - \sqrt{5})/2] \\ \ln \mathfrak{B}(0) = 0 \end{gathered} \tag{2.89}$$

Using (2.88) and (2.89) in (2.86) yields

$$D_1 = D_2 = 0 \tag{2.90}$$

Defining limiting functions of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{A}$, and $\mathfrak{B}$ as in (2.50) and (2.51), we obtain the integral equations

$$\begin{aligned} la_\pm(x) &= \ln[-\tanh \tfrac{1}{2}(x - y_{1\pm} \mp i\varepsilon)] + k * lB_\pm \\ lb_\pm(x) &= -2e^{-x} + \ln[-\tanh \tfrac{1}{2}(x - y_{2\pm} \mp i\varepsilon)] + k * lA_\pm \end{aligned} \tag{2.91}$$

We now repeat the analysis of the last subsection with the term $-\pi$ in (2.57) replaced by $\pm i\ln[-\tanh\frac{1}{2}(y_{1\pm}-y_{2\pm}\mp i\pi/2)]$. In this way we derive the finite-size correction to the eigenvalue in strip 1

$$\begin{aligned}\ln T_1(u) \simeq \frac{1}{N}\Bigg(& e^x\left\{(1-2k_{1+})\pi+\frac{1}{2\pi}\int_{-\infty}^{\infty} dy\,\frac{lA_+(y)}{\sinh(y-y_{1+})}\right.\\ &\left.+\frac{1}{\pi}\int_{-\infty}^{\infty} dy\,e^{-y}lB_+(y)+i\ln\left[-\tanh\frac{1}{2}\left(y_{1+}-y_{2+}-\frac{i\pi}{2}\right)\right]\right\}\\ &+e^{-x}\left\{(1-2k_{1-})\pi+\frac{1}{2\pi}\int_{-\infty}^{\infty} dy\,\frac{lA_-(y)}{\sinh(y-y_{1-})}\right.\\ &\left.+\frac{1}{\pi}\int_{-\infty}^{\infty} dy\,e^{-y}lB_-(y)-i\ln\left[-\tanh\frac{1}{2}\left(y_{1-}-y_{2-}+\frac{i\pi}{2}\right)\right]\right\}\Bigg)\end{aligned} \tag{2.92}$$

Applying the usual manipulations to (2.91) now yields

$$\begin{aligned}&\int_{-\infty}^{\infty} dx\,[la'_\pm(x)\,lA_\pm(x)-la_\pm(x)\,lA'_\pm(x)]\\ &\quad+\int_{-\infty}^{\infty} dx\,[lb'_\pm(x)\,lB_\pm(x)-lb_\pm(x)\,lB'_\pm(x)]\\ &=\int_{-\infty}^{\infty} dx\,\{\ln[-\tanh\tfrac{1}{2}(x-y_1)]'\,lA_\pm(x)-\ln[-\tanh\tfrac{1}{2}(x-y_1)]\,lA'_\pm(x)\}\\ &\quad+\int_{-\infty}^{\infty} dx\,\{\ln[-\tanh\tfrac{1}{2}(x-y_2)]'\,lB_\pm(x)-\ln[-\tanh\tfrac{1}{2}(x-y_2)]\,lB'_\pm(x)\}\\ &\quad+2\int_{-\infty}^{\infty} dx\,e^{-x}[lB_\pm(x)+lB'_\pm(x)]\end{aligned} \tag{2.93}$$

Performing an integration by parts using $\ln[-\tanh(-\infty)]=0$ and $\ln[-\tanh(\infty)]=\pm\pi i$, we thus obtain

$$\begin{aligned}&2\int_{-\infty}^{\infty} dx\,\frac{lA_\pm(x)}{\sinh(x-y_{1\pm})}+4\int_{-\infty}^{\infty} dx\,e^{-x}lB_\pm(x)\\ &=\int_{-\infty}^{\infty} dx\,[la'_\pm(x)\,lA_\pm(x)-la_\pm(x)\,lA'_\pm(x)]\pm\pi i\,lA_\pm(\infty)\\ &\quad+\int_{-\infty}^{\infty} dx\,[lb'_\pm(x)\,lB_\pm(x)-lb_\pm(x)\,lB'_\pm(x)]\pm\pi i\,lB_\pm(\infty)\\ &\quad-2\int_{-\infty}^{\infty} dx\,\frac{lB_\pm(x)}{\sinh(x-y_{2\pm})}\end{aligned} \tag{2.94}$$

Changing the variable of integration x to a and b on the right side of (2.94), we obtain the a integral

$$\begin{aligned}
&\int_{-\infty}^{\infty} dx\,[la'_{\pm}(x)\,lA_{\pm}(x)-la_{\pm}(x)\,lA'_{\pm}(x)]\pm\pi i\,lA_{\pm}(\infty)\\
&\quad=\int_{a(-\infty)}^{a(\infty)} da\left[\frac{\ln(1+a)}{a}-\frac{\ln a}{1+a}\right]\pm\pi i\,lA_{\pm}(\infty)\\
&\quad=\int_{0}^{(1-\sqrt{5})/2} da\left[\frac{\ln(1+a)}{a}-\frac{\ln(-a)}{1+a}\right]\\
&\qquad+\int_{1}^{0} da\left[\frac{\ln(1+a)}{a}-\frac{\ln a}{1+a}\right]
\end{aligned}\tag{2.95}$$

In this last step we have dropped the subscripts $\pm$, specified the correct branches of the logarithms, and used the asymptotics

$$\begin{aligned}
a(-\infty)&=1, & a(\infty)&=(1-\sqrt{5})/2\\
b(-\infty)&=0, & b(\infty)&=(1-\sqrt{5})/2
\end{aligned}\tag{2.96}$$

A similar calculation for the b integral in (2.94) yields

$$\begin{aligned}
&\int_{-\infty}^{\infty} dx\,[lb'_{\pm}(x)\,lB_{\pm}(x)-lb_{\pm}(x)\,lB'_{\pm}(x)]\pm\pi i\,lB_{\pm}(\infty)\\
&\quad=\int_{b(-\infty)}^{b(\infty)} db\left[\frac{\ln(1+b)}{b}-\frac{\ln b}{1+b}\right]\pm\pi i\,lB_{\pm}(\infty)\\
&\quad=\int_{0}^{(1-\sqrt{5})/2} db\left[\frac{\ln(1+b)}{b}-\frac{\ln(-b)}{1+b}\right]
\end{aligned}\tag{2.97}$$

Inserting (2.95), (2.97) into (2.94), we derive

$$\begin{aligned}
&2\int_{-\infty}^{\infty} dx\,\frac{lA_{\pm}(x)}{\sinh(x-y_{1\pm})}+4\int_{-\infty}^{\infty} dx\,e^{-x}lB_{\pm}(x)\\
&\quad=2\int_{0}^{(\sqrt{5}-1)/2} da\left[\frac{\ln(1-a)}{a}+\frac{\ln a}{1-a}\right]-\int_{0}^{1} da\left[\frac{\ln(1+a)}{a}-\frac{\ln a}{1+a}\right]\\
&\qquad-2\int_{-\infty}^{\infty} dx\,\frac{lB_{\pm}(x)}{\sinh(x-y_{2\pm})}\\
&\quad=-4L\left(\frac{\sqrt{5}-1}{2}\right)-2L\left(\frac{1}{2}\right)-2\int_{-\infty}^{\infty} dx\,\frac{lB_{\pm}(x)}{\sinh(x-y_{2\pm})}\\
&\quad=-\frac{17}{10}\frac{\pi^2}{3}-2\int_{-\infty}^{\infty} dx\,\frac{lB_{\pm}(x)}{\sinh(x-y_{2\pm})}
\end{aligned}\tag{2.98}$$

The remaining integral in (2.98) is evaluated analogously to deriving (2.57). It is achieved by remembering the condition $T(u_z - 2\pi/5) = -1$ each zero of $T(u)$ has to satisfy. Applying this requirement to $u_{2\pm}$ in the scaling limit, we find

$$a_\pm\left(y_{2\pm} \mp i\frac{\pi}{2}\right) = -1, \qquad \pm i\, la_\pm\left(y_{2\pm} \mp i\frac{\pi}{2}\right) = (2k_{2\pm} - 1)\pi \quad (2.99)$$

where $k_{2\pm}$ are integers. The lhs of this last equation can be written in terms of the functions $lB_\pm$ using (2.91),

$$\pm i\, la_\pm\left(y_{2\pm} \mp i\frac{\pi}{2}\right) = \pm i \ln\left[-\tanh\frac{1}{2}\left(y_{2\pm} - y_{1\pm} \mp i\frac{\pi}{2}\right)\right] \pm \frac{i}{2\pi}\int_{-\infty}^{\infty} dx\, \frac{lB_\pm(x)}{\cosh(y_{2\pm} \mp i\pi/2 - x)} \quad (2.100)$$

Combining the last two equations yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, \frac{lB_\pm(x)}{\sinh(x - y_{2\pm})} = \mp i \ln\left[-\tanh\frac{1}{2}\left(y_{2\pm} - y_{1\pm} \mp \frac{i\pi}{2}\right)\right] + (2k_{2\pm} - 1)\pi \quad (2.101)$$

Inserting this into (2.92), (2.98), using

$$i \ln\left[-\tanh\frac{1}{2}\left(y_{1\pm} - y_{2\pm} \mp \frac{i\pi}{2}\right)\right] + i \ln\left[-\tanh\frac{1}{2}\left(y_{2\pm} - y_{1\pm} \mp \frac{i\pi}{2}\right)\right] = \mp\pi \quad (2.102)$$

which follows from our choice of branches, we derive the results

$$E = -\ln T_1(u) \simeq \frac{2\pi}{N}\left[\left(k_+ + k_- - \frac{103}{120}\right)\sin 5u + (k_+ - k_-)\, i \cos 5u\right] \quad (2.103)$$

and

$$E - E_0 \simeq \frac{2\pi}{N}\left[\left(k_+ + k_- - \frac{4}{5}\right)\sin 5u + (k_+ - k_-)\, i \cos 5u\right] \quad (2.104)$$

Here $k_+ := k_{1+} + k_{2+}$ and $k_- := k_{1-} + k_{2-}$ are integers giving rise to a tower of eigenvalues of the transfer matrix. The smallest allowed choice

$k_+ = k_- = 1$ gives the eigenvalue corresponding to the leading thermal scaling field with scaling dimension

$$x_4 = 6/5 \tag{2.105}$$

and zero spin.

3. CRITICAL HARD HEXAGONS

The critical hard hexagon model is described by the same transfer matrix as tricritical hard squares. The physical regime, however, given by $-\pi/5 \leqslant u \leqslant 0$ is different. So, although the eigenvalues of the two models are the same, the eigenvalues appear in a different order and different eigenvalues dominate. In particular, the largest eigenvalue for critical hard hexagons exhibits different analyticity properties and analyticity domains from those of tricritical hard squares. Nevertheless, the methods of Section 2 can be applied with suitable modifications. We first treat the largest eigenvalue to obtain the central charge and then generalize our arguments to the next-largest eigenvalues in subsequent subsections. Throughout this section we will assume that $N = 0 \pmod 3$.

3.1. The Central Charge $c = 4/5$

The eigenvalues of critical hard hexagons are characterized by their patterns of zeros in the complex u plane. Using periodicity, we restrict our attention in this case to the strip $-3\pi/5 < \mathrm{Re}(u) \leqslant 2\pi/5$. As for tricritical hard squares, zeros are found to occur on the lines $\mathrm{Re}(u) = -2\pi/5$, $-\pi/5$, $-\pi/10$, $\pi/10$, $3\pi/10$, $2\pi/5$. In this case, however, the zeros of the largest eigenvalues become dense in the thermodynamic limit on the lines $\mathrm{Re}(u) = -2\pi/5$, $\pi/10$. A typical pattern of zeros for the largest eigenvalue $T(u)$ is shown in Fig. 8. We therefore distinguish two strips where the largest eigenvalue $T(u)$ is analytic and nonzero,

$$\begin{aligned} &\text{strip 1:} \qquad -\frac{2\pi}{5} < \mathrm{Re}(u) < \frac{\pi}{10} \\ &\text{strip 2:} \qquad \frac{\pi}{10} < \mathrm{Re}(u) < \frac{3\pi}{5} \end{aligned} \tag{3.1}$$

Again we represent $T(u)$ in these strips by the known bulk behavior multiplied by some correction functions $l(u)$,

$$\begin{aligned} T_1(u) &= z_1(u)^N\, l_1(u), \qquad z_1(u) = \left[\frac{\sin(5u/3 - \pi/3)}{\sin(5u/3 + \pi/3)}\right] \\ T_2(u) &= z_2(u)^N\, l_2(u), \qquad z_2(u) = \left[\frac{\sin(5u/3)}{\sin(5u/3 - 2\pi/3)}\right] \end{aligned} \tag{3.2}$$

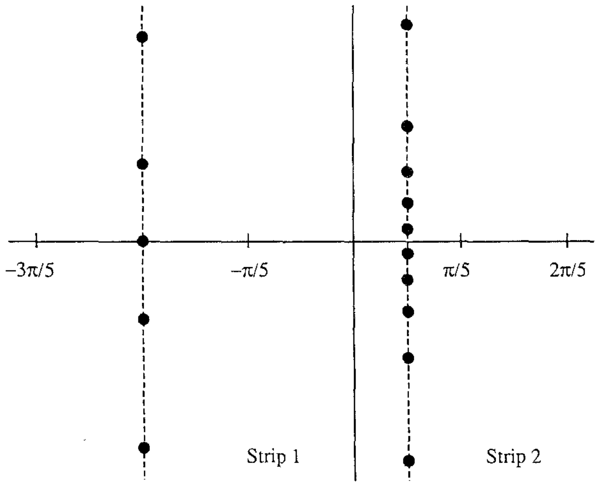

Fig. 8. Schematic representation of the zeros in the complex u plane of the largest eigenvalue of critical hard hexagons. The finite-size correction to this eigenvalue yields the central charge $c=4/5$.

Inserting (3.2) into the functional relation (1.15), we obtain

$$\begin{aligned} l_1(u)\, l_1(u+\lambda)/l_2(u+3\lambda) &= p_2(u) \\ l_2(u)\, l_2(u+\lambda)/l_1(u-2\lambda) &= p_1(u) \end{aligned} \tag{3.3}$$

where the functions on the rhs are defined by

$$\begin{aligned} p_1(u) &:= 1+1/T_1(u-2\lambda) \\ p_2(u) &:= 1+1/T_2(u+3\lambda) \end{aligned} \tag{3.4}$$

Since $l_1(u)$, $l_2(u)$, $p_1(u)$, and $p_2(u)$ are ANZC in their strips of analyticity, the derivatives of $\ln l_1(u)$, etc., can be Fourier-transformed. Adopting the notation of Section 2 for Fourier transforms, it follows from (3.3) that

$$\begin{aligned} (1+e^{\lambda k})\, L_1(k) - e^{3\lambda k} L_2(k) &= P_2(k) \\ (1+e^{\lambda k})\, L_2(k) - e^{-2\lambda k} L_1(k) &= P_1(k) \end{aligned} \tag{3.5}$$

These linear equations are readily solved for $L_1(k)$ and $L_2(k)$,

$$\begin{aligned} L_1(k) &= \frac{e^{3\lambda k}}{1+e^{\lambda k}+e^{2\lambda k}}\, P_1(k) + \frac{1+e^{\lambda k}}{1+e^{\lambda k}+e^{2\lambda k}}\, P_2(k) \\ L_2(k) &= \frac{1+e^{\lambda k}}{1+e^{\lambda k}+e^{2\lambda k}}\, P_1(k) + \frac{e^{-2\lambda k}}{1+e^{\lambda k}+e^{2\lambda k}}\, P_2(k) \end{aligned} \tag{3.6}$$

Transforming back, we find a double integral. Evaluating the k integral using

$$\int_{-\infty}^{\infty} dk \frac{e^{ku}}{1+e^{k\pi/5}+e^{2k\pi/5}} = \frac{10}{\sqrt{3}} \frac{\sin(5u/3+2\pi/3)}{\sin 5u}, \qquad 0<\mathrm{Re}(u)<\frac{2\pi}{5} \tag{3.7}$$

we obtain

$$\begin{aligned}[\ln l_1(u)]' = {} & \frac{1}{2\pi i}\frac{10}{\sqrt{3}} \int_{\mathrm{Re}(v)=\pi/4} dv\, [\ln p_1(v)]' \frac{\sin[\frac{5}{3}(u-v)+2\pi/3]}{\sin 5(u-v)} \\ & + \frac{1}{2\pi i}\frac{10}{\sqrt{3}} \int_{\mathrm{Re}(v)=-\pi/4} dv\, [\ln p_2(v)]' \frac{\sin[\frac{5}{3}(u-v)+\pi/3]}{\sin 5(u-v)}\end{aligned} \tag{3.8}$$

$$\begin{aligned}[\ln l_2(u)]' = {} & \frac{1}{2\pi i}\frac{10}{\sqrt{3}} \int_{\mathrm{Re}(v)=\pi/4} dv\, [\ln p_1(v)]' \frac{\sin[\frac{5}{3}(u-v)+\pi/3]}{\sin 5(u-v)} \\ & + \frac{1}{2\pi i}\frac{10}{\sqrt{3}} \int_{\mathrm{Re}(v)=-\pi/4} dv\, [\ln p_2(v)]' \frac{\sin[\frac{5}{3}(u-v)]}{\sin 5(u-v)}\end{aligned} \tag{3.9}$$

These equations are valid in the strips $-\pi/4<\mathrm{Re}(u)<-\pi/20$ and $\pi/4<\mathrm{Re}(u)<9\pi/20$, respectively.

To make further progress, we restrict the T and p functions to appropriate straight lines and define the following functions of a real variable x:

$$\begin{aligned} \mathfrak{a}(x) &:= 1/T_1\left(\frac{3i}{10}x-\frac{3\pi}{20}\right), \qquad \mathfrak{A}(x) := p_1\left(\frac{3i}{10}x+\frac{\pi}{4}\right)=1+\mathfrak{a}(x) \\ \mathfrak{b}(x) &:= 1/T_2\left(\frac{3i}{10}x+\frac{7\pi}{20}\right), \qquad \mathfrak{B}(x) := p_2\left(\frac{3i}{10}x-\frac{\pi}{4}\right)=1+\mathfrak{b}(x)\end{aligned} \tag{3.10}$$

After integrating (3.9) with respect to u and introducing the functions defined in (3.10), we are led to the integral equations

$$\begin{aligned} \ln \mathfrak{a}(x) &= -\ln z_1(3ix/10-3\pi/20)^N - s*\ln\mathfrak{A} - c*\ln\mathfrak{B} + D_1 \\ \ln \mathfrak{b}(x) &= -\ln z_2(3ix/10+7\pi/20)^N - c*\ln\mathfrak{A} - s*\ln\mathfrak{B} + D_2\end{aligned} \tag{3.11}$$

where the functions s and c of the kernel are given by

$$s(y)=\frac{\sqrt{3}\sinh\frac{1}{2}y}{2\pi\sinh\frac{3}{2}y}, \qquad c(y)=\frac{\overline{\Theta}\cosh\frac{1}{2}y}{\Delta\pi\cosh\frac{3}{2}y} \tag{3.12}$$

and the branches of $\ln z_1$ and $\ln z_2$ are fixed by requiring that they vanish as $x\to\infty$.

We next evaluate the constants D_1 and D_2 from the asymptotic behavior of $T(u)$ for large (positive or negative) imaginary parts of u,

$$\lim_{\mathrm{Im}(u)\to\pm\infty} T(u) = \frac{1+\sqrt{5}}{2} \tag{3.13}$$

Using (3.10), we see that

$$\mathfrak{a}(\infty) = \mathfrak{b}(\infty) = (\sqrt{5}-1)/2 \tag{3.14}$$

From this and the integrals

$$\int_{-\infty}^{\infty} dy\, s(y) = \tfrac{1}{3}, \qquad \int_{-\infty}^{\infty} dy\, c(y) = \tfrac{2}{3} \tag{3.15}$$

we deduce that the constants in the integral equations (3.11) vanish,

$$D_1 = D_2 = 0 \tag{3.16}$$

Now we observe that the functions z_1 and z_2 have the scaling behavior

$$\begin{aligned} &\lim_{N\to\infty} z_1\left(\pm\frac{3i}{10}(x+\ln N)-\frac{3\pi}{20}\right)^N = \exp(\sqrt{3}\,e^{-x}) \\ &\lim_{N\to\infty} z_2\left(\pm\frac{3i}{10}(x+\ln N)+\frac{7\pi}{20}\right)^N = \exp(\sqrt{3}\,e^{-x}) \end{aligned} \tag{3.17}$$

Assuming that $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{A}$, and $\mathfrak{B}$ scale accordingly, we define the limiting functions

$$\begin{aligned} a(x) &:= \lim_{N\to\infty} \mathfrak{a}(x+\ln N), & la(x) &:= \ln a(x) \\ A(x) &:= \lim_{N\to\infty} \mathfrak{A}(x+\ln N) = 1 + a(x), & lA(x) &:= \ln A(x) \\ b(x) &:= \lim_{N\to\infty} \mathfrak{b}(x+\ln N), & lb(x) &:= \ln b(x) \\ B(x) &:= \lim_{N\to\infty} \mathfrak{B}(x+\ln N) = 1 + b(x), & lB(x) &:= \ln B(x) \end{aligned} \tag{3.18}$$

In the case of the largest eigenvalue, the limiting functions in the negative scaling regime are simply related to the above ones because $\mathfrak{a}(-x) = \mathfrak{b}(x)$. For the next-largest eigenvalues, however, this symmetry relation is not valid in general and so in the next subsections we treat the two scaling regimes separately.

From (3.11) we immediately obtain integral equations in the positive scaling regime,

$$\begin{aligned} la(x) &= -\sqrt{3}\, e^{-x} - s * lA - c * lB \\ lb(x) &= -\sqrt{3}\, e^{-x} - c * lA - s * lB \end{aligned} \tag{3.19}$$

As for tricritical hard squares, the matrix K of the kernel

$$K := \begin{pmatrix} s & c \\ c & s \end{pmatrix} \tag{3.20}$$

is symmetric,

$$K^T(y-x) = K(x-y) \tag{3.21}$$

This is essential for the last step in the calculation of the central charge c.

We next consider the largest eigenvalue in the physical strip $T_1(u)$. Using the variable $u = 3ix/10 - 3\pi/20$, applying (3.10) and (3.11), and scaling the integration variable, we derive

$$\begin{aligned} \ln T_1(u) &= -\ln \mathfrak{a}(x) = N \ln z_1(u) + s * \ln \mathfrak{A} + c * \ln \mathfrak{B} \\ &= N \ln z_1(u) + \int_0^\infty dy\, [s(x-y) + c(x+y)] \ln \mathfrak{A}(y) \\ &\quad + \int_0^\infty dy\, [c(x-y) + s(x+y)] \ln \mathfrak{B}(y) \\ &\simeq N \ln z_1(u) + \frac{\sqrt{3}}{2\pi N} (e^x + e^{-x}) \int_{-\infty}^\infty dy\, e^{-y} [lA(y) + lB(y)] \end{aligned} \tag{3.22}$$

The integral in (3.22) can be calculated from (3.19) without explicitly solving this set of integral equations. For this purpose we first take the derivative of (3.19),

$$\begin{aligned} la'(x) &= \sqrt{3}\, e^{-x} - s * lA' - c * lB' \\ lb'(x) &= \sqrt{3}\, e^{-x} - c * lA' - s * lB' \end{aligned} \tag{3.23}$$

Multiplying (3.23) with $lA(x)$ and $lB(x)$ and (3.19) with $lA'(x)$ and $lB'(x)$, subtracting, and integrating, we obtain

$$\begin{aligned} &\int_{-\infty}^\infty dx\, [la'(x)\, lA(x) - la(x)\, lA'(x) + lb'(x)\, lB(x) - lb(x)\, lB'(x)] \\ &\quad = \sqrt{3} \int_{-\infty}^\infty dx\, e^{-x} [lA(x) + lA'(x) + lB(x) + lB'(x)] \end{aligned} \tag{3.24}$$

where the contributions of the kernel K cancel due to the symmetry (3.21). Integrating the rhs by parts and changing the variable of integration x to a and b on the lhs gives

$$\begin{aligned}
&2\sqrt{3}\int_{-\infty}^{\infty} dx\, e^{-x}[lA(x)+lB(x)] \\
&=\int_{-\infty}^{\infty} dx\, [la'(x)\, lA(x)-la(x)\, lA'(x)+lb'(x)\, lB(x)-lb(x)\, lB'(x)] \\
&=\int_{a(-\infty)}^{a(\infty)} da\left[\frac{\ln(1+a)}{a}-\frac{\ln a}{1+a}\right]+\int_{b(-\infty)}^{b(\infty)} db\left[\frac{\ln(1+b)}{b}-\frac{\ln b}{1+b}\right] \\
&=4L_{+}\left(\frac{\sqrt{5}-1}{2}\right)=4L\left(\frac{3-\sqrt{5}}{2}\right)=\frac{4}{5}\frac{\pi^2}{3}
\end{aligned} \tag{3.25}$$

The Rogers dilogarithms are calculated using the asymptotics of $a(x)$ and $b(x)$, which can be read off from (3.14) after recalling the definitions (3.18) and (3.19),

$$\begin{aligned}
a(-\infty)&=0, \qquad a(\infty)=(\sqrt{5}-1)/2 \\
b(-\infty)&=0, \qquad b(\infty)=(\sqrt{5}-1)/2
\end{aligned} \tag{3.26}$$

Collecting the results (3.22) and (3.25) together and observing the relations

$$\cosh x=-\sin\frac{10}{3}u, \qquad \sinh x=-i\cos\frac{10}{3}u \tag{3.27}$$

we obtain

$$E_0=-\ln T_1(u)=-N\ln z_1(u)+\frac{4}{5}\frac{\pi}{6N}\sin\frac{10}{3}u \tag{3.28}$$

from which we see that

$$c=4/5 \tag{3.29}$$

3.2. The Leading Magnetic Scaling Dimension $x=2/15$

The next-largest eigenvalues for critical hard hexagons are described by patterns of zeros with $O(2N/3)$ zeros on the line $\mathrm{Re}(u)=\pi/10$, $O(N/3)$ zeros on the line $\mathrm{Re}(u)=-2\pi/5$, and a finite number of zeros shifted to other lines in the complex plane. The only *a priori* requirement these patterns have to meet is the reflection symmetry about the axis $\mathrm{Re}(u)=\pi/10$.

The patterns for leading excitations turn out to be more diverse than for tricritical hard squares. As input to our calculations we will again assume some basic properties of the next-largest eigenvalues, such as the qualitative pattern of zeros in the complex plane and the corresponding asymptotic behavior of $T(u)$ given by the golden numbers. The two next-largest eigenvalues $T(u)$ we treat in this subsection are complex conjugates with zeros distributed asymmetrically on the lines $\mathrm{Re}(u) = \pi/10$ and $-2\pi/5$ as shown schematically in Fig. 9. These eigenvalues have the same analyticity strips as the largest eigenvalue, but different asymptotics given by

$$\lim_{\mathrm{Im}(u) \to \pm\infty} T(u) = \frac{1-\sqrt{5}}{2} \tag{3.30}$$

The entire analysis of the last subsection can therefore be repeated to derive the integral equations (3.11), where the constants D_1 and D_2 need to be evaluated with the different asymptotic behavior

$$\mathfrak{a}(\infty) = \mathfrak{b}(\infty) = -(1+\sqrt{5})/2 \tag{3.31}$$

The branches of the functions $\ln \mathfrak{a}$ and $\ln \mathfrak{b}$ can be specified by requiring

$$\ln \mathfrak{a}(\infty) = \ln[(1+\sqrt{5})/2] - \pi i, \qquad \ln \mathfrak{b}(\infty) = \ln[(1+\sqrt{5})/2] + \pi i \tag{3.32}$$

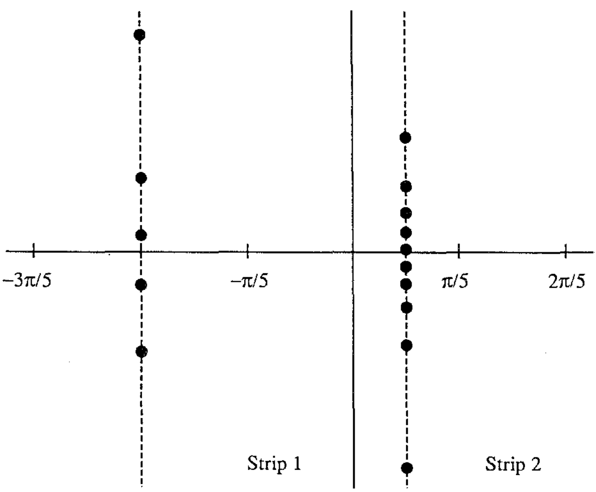

Fig. 9. Schematic representation of the zeros in the complex u plane of the leading magnetic eigenvalue of critical hard hexagons. The finite-size correction to this eigenvalue yields the result $x = 2/15$.

We next pick one of the complex conjugate eigenvalues and check that the branches of the functions $\ln \mathfrak{A}$ and $\ln \mathfrak{B}$ can be fixed such that

$$\begin{aligned} &\ln \mathfrak{A}(\pm\infty)=\ln[(\sqrt{5}-1)/2]-\pi i, &&\ln \mathfrak{B}(\pm\infty)=\ln[(\sqrt{5}-1)/2]+\pi i \\ &\lim_{N\to\infty} \ln \mathfrak{A}(0)=0, &&\lim_{N\to\infty} \ln \mathfrak{B}(0)=0 \end{aligned} \tag{3.33}$$

where the last line is consistent with $\lim_{N\to\infty} \mathfrak{A}(0)=\lim_{N\to\infty} \mathfrak{B}(0)=1$. Using (3.11), (3.32), and (3.33) we therefore deduce that

$$D_1=-2\pi i/3, \qquad D_2=2\pi i/3 \tag{3.34}$$

We next define limiting functions of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{A}$, and $\mathfrak{B}$, in the positive and negative scaling regimes,

$$\begin{aligned} &a_\pm(x):=\lim_{N\to\infty} \mathfrak{a}(\pm x\pm\ln N), && la_\pm(x):=\ln a_\pm(x) \\ &A_\pm(x):=\lim_{N\to\infty} \mathfrak{A}(\pm x\pm\ln N)=1+a_\pm(x), && lA_\pm(x):=\ln A_\pm(x) \\ &b_\pm(x):=\lim_{N\to\infty} \mathfrak{b}(\pm x\pm\ln N), && lb_\pm(x):=\ln b_\pm(x) \\ &B_\pm(x):=\lim_{N\to\infty} \mathfrak{B}(\pm x\pm\ln N)=1+b_\pm(x), && lB_\pm(x):=\ln B_\pm(x) \end{aligned} \tag{3.35}$$

where the branches are fixed such that

$$\begin{aligned} &la_\pm(\infty)=\ln[(1+\sqrt{5})/2]-\pi i, && lb_\pm(\infty)=\ln[(1+\sqrt{5})/2]+\pi i \\ &lA_\pm(\infty)=\ln[(\sqrt{5}-1)/2]-\pi i, && lB_\pm(\infty)=\ln[(\sqrt{5}-1)/2]+\pi i \\ &lA_\pm(-\infty)=0, && lB_\pm(-\infty)=0 \end{aligned} \tag{3.36}$$

From (3.11) we then immediately obtain the integral equations in the scaling regimes

$$\begin{aligned} la_\pm(x)&=-\sqrt{3}\,e^{-x}-s*lA_\pm-c*lB_\pm-2\pi i/3 \\ lb_\pm(x)&=-\sqrt{3}\,e^{-x}-c*lA_\pm-s*lB_\pm+2\pi i/3 \end{aligned} \tag{3.37}$$

We next turn to the eigenvalue in the physical strip $T_1(u)$. Using the variable $u=3ix/10-3\pi/20$, applying (3.10) and (3.11) and scaling the integration variable, we derive

$$\ln T_1(u) = -\ln \mathfrak{a}(x) = N \ln z_1(u) + s * \ln \mathfrak{A} + c * \ln \mathfrak{B} + \frac{2\pi i}{3}$$

$$= N \ln z_1(u) + \frac{2\pi i}{3} + \int_0^\infty dy \, [s(x-y) \ln \mathfrak{A}(y) + s(x+y) \ln \mathfrak{A}(-y)]$$

$$+ \int_0^\infty dy \, [c(x-y) \ln \mathfrak{B}(y) + c(x+y) \ln \mathfrak{B}(-y)]$$

$$\simeq N \ln z_1(u) + \frac{2\pi i}{3} + \frac{\sqrt{3}}{2\pi N} e^x \int_{-\infty}^\infty dy \, e^{-y} [lA_+(y) + lB_+(y)]$$

$$+ \frac{\sqrt{3}}{2\pi N} e^{-x} \int_{-\infty}^\infty dy \, e^{-y} [lA_-(y) + lB_-(y)] \tag{3.38}$$

The integrals in (3.38) can be calculated from (3.37). In order to simplify the notation, we drop the subscripts $\pm$. After multiplying the derivative of (3.37) with $lA(x)$ and $lB(x)$ and (3.37) with lA' and lB' and lastly integrating the difference, we obtain

$$\int_{-\infty}^\infty dx \, [la'(x) \, lA(x) - la(x) \, lA'(x) + lb'(x) \, lB(x) - lb(x) \, lB'(x)]$$

$$- \frac{2\pi i}{3} [lA(\infty) - lB(\infty)]$$

$$= \sqrt{3} \int_{-\infty}^\infty dx \, e^{-x} [lA(x) + lA'(x) + lB(x) + lB'(x)] \tag{3.39}$$

Integrating the rhs by parts gives

$$2\sqrt{3} \int_{-\infty}^\infty dx \, e^{-x} [lA(x) + lB(x)]$$

$$= -\frac{4\pi^2}{3} + \int_{-\infty}^\infty dx \, [la'(x) lA(x) - la(x) lA'(x) + lb'(x) lB(x) - lb(x) lB'(x)] \tag{3.40}$$

Changing the variable of integration x to a and b on the rhs then yields

$$\int_{-\infty}^\infty dx \, [la'(x) \, lA(x) - la(x) \, lA'(x) + lb'(x) \, lB(x) - lb(x) \, lB'(x)]$$

$$= \int_{a(-\infty)}^{a(\infty)} da \left[\frac{\ln(1+a)}{a} - \frac{\ln a}{1+a} \right] + \int_{b(-\infty)}^{b(\infty)} db \left[\frac{\ln(1+b)}{b} - \frac{\ln b}{1+b} \right] \tag{3.41}$$

where the terminals can be read off from (3.36) and (3.31) after recalling the definitions (3.35),

$$\begin{aligned} a(-\infty)&=0, \qquad a(\infty)= -(1+\sqrt{5})/2 \\ b(-\infty)&=0, \qquad b(\infty)= -(1+\sqrt{5})/2 \end{aligned} \tag{3.42}$$

The last integrals can be calculated using the branches specified in (3.36). Appealing to Cauchy's theorem, we also take the a and b integration paths along the real axis from 0 to $-(1+\sqrt{5})/2$ surrounding the point -1 in the lower and upper half-planes, respectively, as shown in Fig. 10. Thus, we trace back the integrals in (3.41) to integrals of real functions,

$$\int_{a(-\infty)}^{a(\infty)} da\,\frac{\ln(1+a)}{a}=\int_0^{-1} dy\,\frac{\ln(1+y)}{y}+\int_{-1}^{-(1+\sqrt{5})/2} dy\,\frac{\ln[-(1+y)]-\pi i}{y}$$

$$\int_{a(-\infty)}^{a(\infty)} da\,\frac{\ln a}{1+a}=\int_0^{-(1+\sqrt{5})/2} da\,\frac{\ln(-a)-\pi i}{1+a}$$

$$=\int_0^{-(1+\sqrt{5})/2} dy\frac{\ln(-y)}{1+y}-\pi^2-\pi i\ln\frac{\sqrt{5}-1}{2}$$

$$\int_{b(-\infty)}^{b(\infty)} db\,\frac{\ln(1+b)}{b}=\int_0^{-1} dy\,\frac{\ln(1+y)}{y}+\int_{-1}^{-(1+\sqrt{5})/2} dy\,\frac{\ln[-(1+y)]+\pi i}{y}$$

$$\int_{b(-\infty)}^{b(\infty)} db\,\frac{\ln b}{1+b}=\int_0^{-(1+\sqrt{5})/2} db\,\frac{\ln(-b)+\pi i}{1+b}$$

$$=\int_0^{-(1+\sqrt{5})/2} dy\,\frac{\ln(-y)}{1+y}-\pi^2+\pi i\ln\frac{\sqrt{5}-1}{2} \tag{3.43}$$

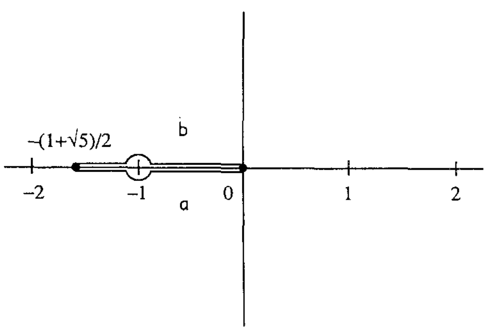

Fig. 10. The lower and upper paths from 0 to $-(1+\sqrt{5})/2$ are the paths in the complex a and b planes respectively needed to evaluate the integrals leading to the scaling dimension $x=2/15$. A semicircular path is taken around the pole at -1.

Collecting (3.41) and (3.43) together gives

$$\int_{-\infty}^{\infty} dx\ [la'(x)\, lA(x) - la(x)\, lA'(x) + lb'(x)\, lB(x) - lb(x)\, lB'(x)]$$

$$= 2\int_0^1 dy \left[\frac{\ln y}{1-y} + \frac{\ln(1-y)}{y}\right] + 2\int_1^{(1+\sqrt{5})/2} dy \left[\frac{\ln(y-1)}{y} - \frac{\ln y}{y-1}\right] + 2\pi^2$$

$$= -4L(1) - 4L\left(\frac{3-\sqrt{5}}{2}\right) + 2\pi^2 = \frac{16}{15}\pi^2 \tag{3.44}$$

where we have evaluated the Rogers dilogarithms. Using (3.38) and (3.40), we thus obtain the result

$$E = -\ln T_1(u) = -N \ln z_1(u) - \frac{2\pi i}{3} - \frac{4}{5}\frac{\pi}{6N} \sin\frac{10}{3}u \tag{3.45}$$

and

$$E - E_0 = -\frac{2\pi i}{3} - \frac{2}{15}\frac{2\pi}{N} \sin\frac{10}{3}u \tag{3.46}$$

Comparing with (1.25), we see that

$$x_1 = x_2 = 2/15 \tag{3.47}$$

with zero spin.

3.3. The Leading Thermal Scaling Dimension $x = 4/5$

The eigenvalues $T(u)$ we consider in this subsection are characterized by four shifted zeros. Two of them, $u_{1\pm}$, are located close to the line $\mathrm{Re}(u) = -\pi/5$. The other two zeros, $u_{2\pm}$, are located close to the line $\mathrm{Re}(u) = 2\pi/5$,

$$\begin{aligned} u_{1+} &= \frac{3i}{10}(y_{1+} + \ln N) - \frac{2\pi}{10}, & u_{2+} &= \frac{3i}{10}(y_{2+} + \ln N) + \frac{4\pi}{10} \\ u_{1-} &= -\frac{3i}{10}(y_{1-} + \ln N) - \frac{2\pi}{10}, & u_{2-} &= -\frac{3i}{10}(y_{2-} + \ln N) + \frac{4\pi}{10} \end{aligned} \tag{3.48}$$

Here y_{1+}, y_{2+} and y_{1-}, y_{2-} are complex conjugate pairs as shown in Fig. 11. All four of these zeros deviate off the lines $\mathrm{Re}(u) = -\pi/5, 2\pi/5$ toward the imaginary axis. We represent $T(u)$ in strips 1 and 2 by explicitly taking into account the zeros $u_{1\pm}$ and $u_{2\pm}$,

$$\begin{aligned} T_1(u) &= z_1(u)^N f(u) \cdot l_1(u) \\ T_2(u) &= z_2(u)^N g(u) \cdot l_2(u) \end{aligned} \tag{3.49}$$

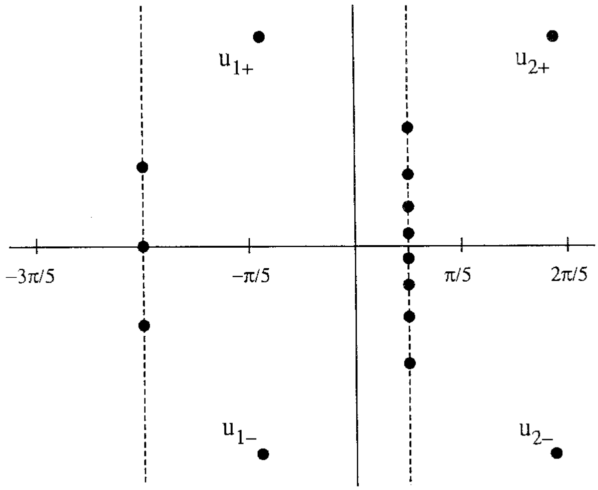

Fig. 11. Schematic representation of the zeros in the complex u plane of the leading thermal eigenvalue of critical hard hexagons. The finite-size correction to this eigenvalue yields the scaling dimension $x = 4/5$.

where

$$\begin{aligned} f(u) &:= z_1(u - u_{1+} + \pi/5)\, z_1(u - u_{1-} + \pi/5)\, z_1(u - u_{2+})\, z_1(u - u_{2-}) \\ g(u) &:= 1/f(u - \pi/5) \end{aligned} \tag{3.50}$$

By inspection we see that l_1 and l_2 are ANZC in strips somewhat narrower than strips 1 and 2 given by (3.1). Because of the functional relation

$$f(u)\, f(u + \pi/5)\, f(u + 2\pi/5) = 1 \tag{3.51}$$

we obtain the same equations as in (3.3) and (3.4). Hence we may carry through the analysis up to (3.10) as before. Instead of (3.11) we now get

$$\begin{aligned} \ln \mathfrak{a}(x) &= -\ln z_1(3ix/10 - 3\pi/20)^N - lf(x) - s * \ln \mathfrak{A} - c * \ln \mathfrak{B} + D_1 \\ \ln \mathfrak{b}(x) &= -\ln z_2(3ix/10 + 7\pi/20)^N - lg(x) - c * \ln \mathfrak{A} - s * \ln \mathfrak{B} + D_2 \end{aligned} \tag{3.52}$$

where lf and lg are defined by

$$lf(x) := \ln f(3ix/10 - 3\pi/20), \qquad lg(x) := \ln g(3ix/10 + 7\pi/20) \tag{3.53}$$

and the branches are specified by requiring

$$lf(\infty) = \frac{8\pi}{3}\, i, \qquad lg(\infty) = -\frac{8\pi}{3}\, i \tag{3.54}$$

The eigenvalues under consideration have asymptotics given by

$$\lim_{\mathrm{Im}(u) \to \pm\infty} T(u) = \frac{1-\sqrt{5}}{2} \tag{3.55}$$

or, in terms of the functions $\mathfrak{a}$ and $\mathfrak{b}$,

$$\mathfrak{a}(\infty) = \mathfrak{b}(\infty) = -(1+\sqrt{5})/2 \tag{3.56}$$

The branches of the functions $\ln \mathfrak{a}$ and $\ln \mathfrak{b}$ can be specified by requiring

$$\ln \mathfrak{a}(\infty) = \ln[(1+\sqrt{5})/2] - \pi i, \qquad \ln \mathfrak{b}(\infty) = \ln[(1+\sqrt{5})/2] + \pi i \tag{3.57}$$

The branches of the functions $\ln \mathfrak{A}$ and $\ln \mathfrak{B}$ can then be fixed such that

$$\begin{aligned} &\ln \mathfrak{A}(\pm\infty) = \ln[(\sqrt{5}-1)/2] \mp \pi i, && \ln \mathfrak{B}(\pm\infty) = \ln[(\sqrt{5}-1)/2] \pm \pi i \\ &\lim_{N\to\infty} \ln \mathfrak{A}(0) = 0, && \lim_{N\to\infty} \ln \mathfrak{B}(0) = 0 \end{aligned} \tag{3.58}$$

From (3.52), (3.57), and (3.58) we therefore deduce that

$$D_1 = 2\pi i, \qquad D_2 = -2\pi i \tag{3.59}$$

We next define limiting functions of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{A}$, and $\mathfrak{B}$ in the positive and negative scaling regimes as in (3.35), where the branches are fixed such that

$$\begin{aligned} &la_\pm(\infty) = \ln[(1+\sqrt{5})/2] \mp \pi i, && lb_\pm(\infty) = \ln[(1+\sqrt{5})/2] \pm \pi i \\ &lA_\pm(\infty) = \ln[(\sqrt{5}-1)/2] \mp \pi i, && lB_\pm(\infty) = \ln[(\sqrt{5}-1)/2] \pm \pi i \\ &lA_\pm(-\infty) = 0, && lB_\pm(-\infty) = 0 \end{aligned} \tag{3.60}$$

From (3.52) we similarly obtain the integral equations

$$\begin{aligned} la_\pm(x) &= -\sqrt{3}\, e^{-x} - lf_\pm(x) - s * lA_\pm - c * lB_\pm \pm 2\pi i/3 \\ lb_\pm(x) &= -\sqrt{3}\, e^{-x} - lg_\pm(x) - c * lA_\pm - s * lB_\pm \mp 2\pi i/3 \end{aligned} \tag{3.61}$$

where $lf_\pm$ and $lg_\pm$ are the scaling limits of lf and lg apart from trivial constants,

$$\begin{aligned} lf_\pm(x) &= \pm\ln\left[z_1\left(\frac{3i}{10}(x-y_{1\pm}) \pm \frac{\pi}{4}\right) z_1\left(\frac{3i}{10}(x-y_{2\pm}) \pm \frac{\pi}{20}\right)\right] \\ lg_\pm(x) &= \mp\ln\left[z_1\left(\frac{3i}{10}(x-y_{1\pm}) \mp \frac{\pi}{20}\right) z_1\left(\frac{3i}{10}(x-y_{2\pm}) \mp \frac{\pi}{4}\right)\right] \end{aligned} \tag{3.62}$$

and the branches are specified by requiring

$$lf_{\pm}(\infty) = \pm\frac{4\pi}{3}i, \qquad lg_{\pm}(\infty) = \mp\frac{4\pi}{3}i \tag{3.63}$$

At this point we turn to the eigenvalue in the physical strip. The function lf in (3.52) contributes a $1/N$ term for $\ln T_1$,

$$lf(x) \simeq \frac{e^x}{N}\sqrt{3}(e^{2\pi i/3 - y_{1+}} + e^{-2\pi i/3 - y_{2+}}) + \frac{e^{-x}}{N}\sqrt{3}(e^{-2\pi i/3 - y_{1-}} + e^{2\pi i/3 - y_{2-}}) \tag{3.64}$$

Another $O(1/N)$ correction term is given by the integrals in (3.52). Collecting these correction terms together yields

$$\begin{aligned}\ln T_1(u) &= -\ln \mathfrak{a}(x) = N\ln z_1(u) + lf(x) + s * \ln \mathfrak{A} + c * \ln \mathfrak{B} - 2\pi i \\ &\simeq N \ln z_1(u) - 2\pi i \\ &\quad + \frac{e^x}{N}\left\{\frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-y}[lA_+(y) + lB_+(y)]\right. \\ &\quad \left. + \sqrt{3}(e^{2\pi i/3 - y_{1+}} + e^{-2\pi i/3 - y_{2+}})\right\} \\ &\quad + \frac{e^{-x}}{N}\left\{\frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-y}[lA_-(y) + lB_-(y)]\right. \\ &\quad \left. + \sqrt{3}(e^{-2\pi i/3 - y_{1-}} + e^{2\pi i/3 - y_{2-}})\right\}\end{aligned} \tag{3.65}$$

To apply our manipulation of the integral equations we have to express the exponentials in (3.65) by integrals involving the functions $lA_{\pm}$ and $lB_{\pm}$. This is achieved by remembering the condition $T(u_z + 3\pi/5) = -1$ that each zero of $T(u)$ must satisfy. Applying this requirement in the scaling limit to u_{1+}, we obtain

$$b_{\pm}\left(y_{1\pm} \mp i\frac{\pi}{6}\right) = -1, \qquad i\, lb_{\pm}\left(y_{1\pm} \mp i\frac{\pi}{6}\right) = \pm(2k_{\pm} - 1)\pi \tag{3.66}$$

where $k_{\pm}$ are integers. Similarly, we find

$$a_{\pm}\left(y_{2\pm} \pm i\frac{\pi}{6}\right) = -1, \qquad i\, la_{\pm}\left(y_{2\pm} \pm i\frac{\pi}{6}\right) = \mp(2k_{\pm} - 1)\pi \tag{3.67}$$

with the same integers $k_\pm$ due to the crossing symmetry $\overline{la(\bar{z})} = lb(z)$. Inserting this into (3.61) gives

$$\begin{aligned}\sqrt{3}(e^{\pm 2\pi i/3 - y_{1\pm}} + e^{\mp 2\pi i/3 - y_{2\pm}})\\ = -(2k_\pm - 1)2\pi + 4\pi/3\\ \mp i(c * lA_\pm + s * lB_\pm)(y_{1\pm} \mp \pi i/6) \pm i(s * lA_\pm + c * lB_\pm)(y_{2\pm} \pm \pi i/6)\\ \pm i[(lf_\pm(y_{2\pm} \pm \pi i/6) - lg_\pm(y_{1\pm} \mp \pi i/6)]\end{aligned} \tag{3.68}$$

where on close inspection the last line is equal to -4π. Hence

$$\begin{aligned}\frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-y}[lA_\pm(y) + lB_\pm(y)] + \sqrt{3}(e^{\pm 2\pi i/3 - y_{1\pm}} + e^{\mp 2\pi i/3 - y_{2\pm}})\\ = \frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-y}[lA_\pm(y) + lB_\pm(y)] - (2k_\pm + 1)2\pi + \frac{4\pi}{3}\\ \mp i(c * lA_\pm + s * lB_\pm)\left(y_{1\pm} \mp \frac{\pi i}{6}\right)\\ \pm i(s * lA_\pm + c * lB_\pm)\left(y_{2\pm} \pm \frac{\pi i}{6}\right)\end{aligned} \tag{3.69}$$

We next manipulate (3.61) to obtain

$$\begin{aligned}\int_{-\infty}^{\infty} dx\,[la'_\pm(x)\, lA_\pm(x) - la_\pm(x)\, lA'_\pm(x)\\ + lb'_\pm(x)\, lB_\pm(x) - lb_\pm(x)\, lB'_\pm(x)]\\ + \int_{-\infty}^{\infty} dx\,[lf'_\pm(x)\, lA_\pm(x) - (lf_\pm(x) \mp 2\pi i/3)\, lA'_\pm(x)]\\ + \int_{-\infty}^{\infty} dx\,[lg'_\pm(x)\, lB_\pm(x) - (lg_\pm(x) \pm 2\pi i/3)\, lB'_\pm(x)]\\ = \sqrt{3}\int_{-\infty}^{\infty} dx\, e^{-x}[lA_\pm(x) + lA'_\pm(x) + lB_\pm(x) + lB'_\pm(x)]\end{aligned} \tag{3.70}$$

Integrating by parts and using

$$\begin{aligned}lf'_\pm(x) &= \pm 2\pi i[c(x - y_{1\pm} \pm \pi i/6) - s(x - y_{2\pm} \mp \pi i/6)]\\ lg'_\pm(x) &= \pm 2\pi i[s(x - y_{1\pm} \pm \pi i/6) - c(x - y_{2\pm} \mp \pi i/6)]\end{aligned} \tag{3.71}$$

we find

$$\frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-y}[lA_{\pm}(y)+lB_{\pm}(y)]$$

$$=\frac{1}{4\pi}\int_{-\infty}^{\infty} dx\, [la'_{\pm}(x)\, lA_{\pm}(x)-la_{\pm}(x)\, lA'_{\pm}(x)$$

$$+lb'_{\pm}(x)\, lB_{\pm}(x)-lb_{\pm}(x)\, lB'_{\pm}(x)]$$

$$-\frac{\pi}{3}\pm i(c*lA_{\pm}+s*lB_{\pm})\left(y_{1\pm}\mp\frac{\pi i}{6}\right)$$

$$\mp i(s*lA_{\pm}+c*lB_{\pm})\left(y_{2\pm}\pm\frac{\pi i}{6}\right) \tag{3.72}$$

where the last integral was already encountered and calculated in the previous subsection. Inserting (3.69) and (3.72) into (3.65), this simplifies to give

$$\ln T_1(u)\simeq N\ln z_1(u)-2\pi i$$

$$+\frac{e^{x}}{N}\left\{\frac{1}{4\pi}\int_{-\infty}^{\infty} dx\, [la'_{+}(x)\, lA_{+}(x)-la_{+}(x)\, lA'_{+}(x)\right.$$

$$\left.+lb'_{+}(x)\, lB_{+}(x)-lb_{+}(x)\, lB'_{+}(x)]-(2k_{+}+1)2\pi+\pi\right\}$$

$$+\frac{e^{-x}}{N}\left\{\frac{1}{4\pi}\int_{-\infty}^{\infty} dx\, [la'_{-}(x)\, lA_{-}(x)-la_{-}(x)\, lA'_{-}(x)\right.$$

$$\left.+lb'_{-}(x)\, lB_{-}(x)-lb_{-}(x)\, B'_{-}(x)]-(2k_{-}+1)2\pi+\pi\right\}$$

$$=N\ln z_1(u)-2\pi i+\frac{e^{x}}{N}\left(-\frac{11}{15}\pi-4\pi k_{+}\right)+\frac{e^{-x}}{N}\left(-\frac{11}{15}\pi-4\pi k_{-}\right)$$

$$=N\ln z_1(u)-2\pi i+\frac{\cosh x}{N}\left(-\frac{11}{15}-2k_{+}-2k_{-}\right)2\pi$$

$$+\frac{\sinh x}{N}(2k_{-}-2k_{+})2\pi \tag{3.73}$$

where the result (3.44) for the integral was used in the last step and k_+ and k_- are integers. We thus obtain the results

$$E\simeq -N\ln z_1(u)-\left(\frac{11}{15}+2k_{+}+2k_{-}\right)\frac{2\pi}{N}\sin\frac{10}{3}u-i(2k_{+}-2k_{-})\frac{2\pi}{N}\cos\frac{10}{3}u \tag{3.74}$$

and

$$E-E_0 \simeq -\left(\frac{4}{5}+2k_+ +2k_-\right)\frac{2\pi}{N}\sin\frac{10}{3}u - i(2k_+ -2k_-)\frac{2\pi}{N}\cos\frac{10}{3}u \tag{3.75}$$

So comparison with (1.25) gives

$$x_3 = 4/5 \tag{3.76}$$

with zero spin. Notice that Eq. (3.75) gives a semitower on x_3 and not the complete tower.

3.4. The Scaling Dimension $x = 17/15$

The eigenvalues $T(u)$ considered in this subsection are characterized by two shifted' zeros u_1 and u_2 which are located close to the lines $\mathrm{Re}(u) = -\pi/10$ and $\mathrm{Re}(u) = 3\pi/10$. The corresponding excitations turn out to belong to the tower above $x_1 = x_2 = 2/15$. Here we assume that u_1 and u_2 lie in the upper half-plane,

$$u_1 = \frac{3i}{10}(y_1 + \ln N) - \frac{\pi}{10}, \qquad u_2 = \frac{3i}{10}(y_2 + \ln N) + \frac{3\pi}{10} \tag{3.77}$$

where y_1, y_2 are complex conjugates as shown in Fig. 12. The two zeros deviate off the lines $\mathrm{Re}(u) = -\pi/10$, $3\pi/10$ away from the imaginary axis.

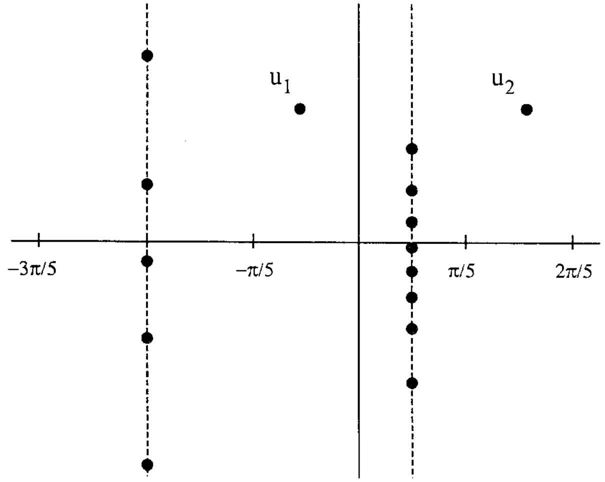

Fig. 12. Schematic representation of the zeros in the complex u plane of an eigenvalue in the same tower as the leading magnetic eigenvalue of critical hard hexagons. The finite-size correction to this eigenvalue yields the result $x = 17/15$.

Eigenvalues corresponding to u_1 and u_2 in the lower half-plane can be obtained by complex conjugation. We represent $T(u)$ in strips 1 and 2 by explicitly taking into account the zeros u_1 and u_2,

$$\begin{aligned} T_1(u) &= z_1(u)^N f(u) \cdot l_1(u) \\ T_2(u) &= z_2(u)^N g(u) \cdot l_2(u) \end{aligned} \tag{3.78}$$

where

$$\begin{aligned} f(u) &:= 1/[z_1(u-u_1-\pi/5)\, z_1(u-u_2+2\pi/5)] \\ g(u) &:= 1/f(u-\pi/5) \end{aligned} \tag{3.79}$$

By inspection we see that l_1 and l_2 are ANZC in strips somewhat narrower than strips 1 and 2 given by (3.1). Proceeding as in the last subsection, we derive the integral equations

$$\begin{aligned} \ln \mathfrak{a}(x) &= -\ln z_1(3ix/10 - 3\pi/20)^N - lf(x) - s * \ln \mathfrak{A} - c * \ln \mathfrak{B} + D_1 \\ \ln \mathfrak{b}(x) &= -\ln z_2(3ix/10 + 7\pi/20)^N - lg(x) - c * \ln \mathfrak{A} - s * \ln \mathfrak{B} + D_2 \end{aligned} \tag{3.80}$$

where lf and lg are defined by

$$lf(x) := \ln f(3ix/10 - 3\pi/20), \qquad lg(x) := \ln g(3ix/10 + 7\pi/20) \tag{3.81}$$

and the branches are specified by requiring

$$lf(\infty) = -\frac{4\pi i}{3}, \qquad lg(\infty) = \frac{4\pi i}{3} \tag{3.82}$$

The eigenvalues at hand have asymptotics given by

$$\lim_{\mathrm{Im}(u) \to \pm\infty} T(u) = \frac{1-\sqrt{5}}{2} \tag{3.83}$$

or, in terms of the functions $\mathfrak{a}$ and $\mathfrak{b}$,

$$\mathfrak{a}(\infty) = \mathfrak{b}(\infty) = -(1+\sqrt{5})/2 \tag{3.84}$$

The branches of the functions $\ln \mathfrak{a}$ and $\ln \mathfrak{b}$ can be specified by requiring

$$\ln \mathfrak{a}(\infty) = \ln[(1+\sqrt{5})/2] - \pi i, \qquad \ln \mathfrak{b}(\infty) = \ln[(1+\sqrt{5})/2] + \pi i \tag{3.85}$$

Similarly, the branches of the functions $\ln \mathfrak{A}$ and $\ln \mathfrak{B}$ can be fixed such that

$$\begin{aligned} &\ln \mathfrak{A}(\pm\infty) = \ln[(\sqrt{5}-1)/2] \mp \pi i, \qquad &&\ln \mathfrak{B}(\pm\infty) = \ln[(\sqrt{5}-1)/2] \pm \pi i \\ &\lim_{N\to\infty} \ln \mathfrak{A}(0) = 0, &&\lim_{N\to\infty} \ln \mathfrak{B}(0) = 0 \end{aligned} \tag{3.86}$$

Using (3.80), (3.85), and (3.86), we deduce

$$D_1 = -2\pi i, \qquad D_2 = 2\pi i \tag{3.87}$$

We next define limiting functions of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{A}$, and $\mathfrak{B}$ in the positive and negative scaling regimes as before and fix the branches as in (3.60). From (3.80) we obtain the integral equations in the positive and negative scaling regimes,

$$\begin{aligned} la_+(x) &= -\sqrt{3}\, e^{-x} - lf_+(x) - s * lA_+ - c * lB_+ - 2\pi i \\ lb_+(x) &= -\sqrt{3}\, e^{-x} - lg_+(x) - c * lA_+ - s * lB_+ + 2\pi i \end{aligned} \tag{3.88}$$

and

$$\begin{aligned} la_-(x) &= -\sqrt{3}\, e^{-x} - s * lA_- - c * lB_- + 2\pi i/3 \\ lb_-(x) &= -\sqrt{3}\, e^{-x} - c * lA_- - s * lB_- - 2\pi i/3 \end{aligned} \tag{3.89}$$

where lf_+ and lg_+ are the scaling limits of lf and lg apart from trivial constants,

$$\begin{aligned} lf_+(x) &= -\ln\left[z_1\left(\frac{3i}{10}(x-y_1)-\frac{\pi}{4}\right) z_1\left(\frac{3i}{10}(x-y_2)-\frac{\pi}{20}\right)\right] \\ lg_+(x) &= +\ln\left[z_1\left(\frac{3i}{10}(x-y_1)+\frac{\pi}{20}\right) z_1\left(\frac{3i}{10}(x-y_2)+\frac{\pi}{4}\right)\right] \end{aligned} \tag{3.90}$$

and the branches are specified by requiring

$$lf_+(\infty) = -\frac{4\pi}{3} i, \qquad lg_+(\infty) = \frac{4\pi}{3} i \tag{3.91}$$

As before, we now consider the eigenvalue in the physical strip. The function lf in (3.80) contributes a $1/N$ term for $\ln T_1$,

$$lf(x) \simeq -\frac{2\pi i}{3} - \frac{e^x}{N}\sqrt{3}(e^{\pi i/3 - y_1} + e^{-\pi i/3 - y_2}) \tag{3.92}$$

Another $O(1/N)$ correction term is given by the integrals in (3.80). Collecting all correction terms together yields

$$\begin{aligned} \ln T_1(u) &= -\ln \mathfrak{a}(x) = N \ln z_1(u) + lf(x) + s * \ln \mathfrak{A} + c * \ln \mathfrak{B} + 2\pi i \\ &\simeq N \ln z_1(u) + 4\pi i/3 \\ &\quad + \frac{e^x}{N}\left\{\frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-y}[lA_+(y) + lB_+(y)] \right. \\ &\quad \left. - \sqrt{3}(e^{\pi i/3 - y_1} + e^{-\pi i/3 - y_2})\right\} \\ &\quad + \frac{e^{-x}}{N}\left\{\frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-y}[lA_-(y) + lB_-(y)]\right\} \end{aligned} \tag{3.93}$$

We next calculate the first integral in (3.93). We need to express the exponentials in (3.93) by integrals involving the functions lA_+ and lB_+. This is achieved by remembering the condition $T(u_z+2\pi/5)=-1$ that each zero of $T(u)$ must satisfy. Applying this requirement to u_1, we get in the scaling limit

$$b_+\left(y_1+i\frac{\pi}{6}\right)=-1, \qquad i\,lb_+\left(y_1+i\frac{\pi}{6}\right)=-(2k-1)\pi \tag{3.94}$$

where k is an integer. Similarly, we find

$$a_+\left(y_2-i\frac{\pi}{6}\right)=-1, \qquad i\,la_+\left(y_2-i\frac{\pi}{6}\right)=(2k-1)\pi \tag{3.95}$$

with the same integer k due to the relation $\overline{la(\bar{z})}=lb(z)$. Inserting this into (3.88) leads to

$$\begin{aligned}
&-\sqrt{3}(e^{\pi i/3-y_1}+e^{-\pi i/3-y_2})\\
&\quad=-(2k-1)2\pi+4\pi\\
&\qquad+i(c*lA_++s*lB_+)(y_1+\pi i/6)-i(s*lA_++c*lB_+)(y_2-\pi i/6)\\
&\qquad-i[lf_+(y_2-\pi i/6)-lg_+(y_1+\pi i/6)]
\end{aligned} \tag{3.96}$$

where on close inspection the last line is equal to -4π. Hence

$$\begin{aligned}
&\frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-y}[lA_+(y)+lB_+(y)]-\sqrt{3}(e^{\pi i/3-y_1}+e^{-\pi i/3-y_2})\\
&\quad=\frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-y}[lA_+(y)+lB_+(y)]-(2k-1)2\pi\\
&\qquad+i(c*lA_++s*lB_+)(y_1+\pi i/6)-i(s*lA_++c*lB_+)(y_2-\pi i/6)
\end{aligned} \tag{3.97}$$

We next manipulate (3.88) to obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} dx\,[la'_+(x)\,lA_+(x)-la_+(x)\,lA'_+(x)\\
&\qquad+lb'_+(x)\,lB_+(x)-lb_+(x)\,lB'_+(x)]\\
&\qquad+\int_{-\infty}^{\infty} dx\,[lf'_+(x)\,lA_+(x)-(lf_+(x)+2\pi i)\,lA'_+(x)]\\
&\qquad+\int_{-\infty}^{\infty} dx\,[lg'_+(x)\,lB_+(x)-(lg_+(x)-2\pi i)\,lB'_+(x)]\\
&\quad=\sqrt{3}\int_{-\infty}^{\infty} dx\, e^{-x}[lA_+(x)+lA'_+(x)+lB_+(x)+lB'_+(x)]
\end{aligned} \tag{3.98}$$

Integrating by parts and using

$$
\begin{aligned}
lf'_+(x) &= -2\pi i[c(x-y_1-\pi i/6)-s(x-y_2+\pi i/6)] \\
lg'_+(x) &= -2\pi i[s(x-y_1-\pi i/6)-c(x-y_2+\pi i/6)]
\end{aligned} \tag{3.99}
$$

we obtain

$$
\begin{aligned}
&\frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-y}[lA_+(y)+lB_+(y)] \\
&\quad = \frac{1}{4\pi}\int_{-\infty}^{\infty} dx\, [la'_+(x)\, lA_+(x) - la_+(x)\, lA'_+(x) \\
&\qquad + lb'_+(x)\, lB_+(x) - lb_+(x)\, lB'_+(x)] \\
&\qquad -\frac{\pi}{3} - i(c*lA_+ + s*lB_+)\left(y_1+\frac{\pi i}{6}\right) \\
&\qquad + i(s*lA_+ + c*lB_+)\left(y_2-\frac{\pi i}{6}\right)
\end{aligned} \tag{3.100}
$$

Inserting (3.97) and (3.100) into (3.93), and manipulating the second integral in (3.93) as in Section 2.2, we obtain

$$
\begin{aligned}
\ln T_1(u) \simeq{}& N\ln z_1(u) + \frac{4\pi i}{3} \\
& + \frac{e^x}{N}\left\{\frac{1}{4\pi}\int_{-\infty}^{\infty} dx\, [la'_+(x)\, lA_+(x) - la_+(x)\, lA'_+(x)\right. \\
& \left. + lb'_+(x)\, lB_+(x) - lb_+(x)\, lB'_+(x)] - (2k-1)2\pi - \frac{\pi}{3}\right\} \\
& + \frac{e^{-x}}{N}\left\{\frac{1}{4\pi}\int_{-\infty}^{\infty} dx\, [la'_-(x)\, lA_-(x) - la_-(x)\, lA'_-(x)\right. \\
& \left. + lb'_-(x)\, lB_-(x) - lb_-(x)\, lB'_-(x)] - \frac{\pi}{3}\right\} \\
={}& N\ln z_1(u) + \frac{4\pi i}{3} + \frac{e^x}{N}\left(\frac{29}{15}\pi - 4\pi k\right) + \frac{e^{-x}}{N}\left(-\frac{1}{15}\pi\right) \\
={}& N\ln z_1(u) + \frac{4\pi i}{3} + \frac{\cosh x}{N}\left(\frac{14}{15} - 2k\right)2\pi + \frac{\sinh x}{N}(1-2k)2\pi
\end{aligned} \tag{3.101}
$$

where we have used the result (3.44) for the integrals in the last step. We thus obtain the results

$$E = -\ln T_1(u) \simeq -N \ln z_1(u) + \frac{2\pi i}{3} - \left(-\frac{14}{15} + 2k\right)\frac{2\pi}{N}\sin\frac{10}{3}u + i(1-2k)\frac{2\pi}{N}\cos\frac{10}{3}u \tag{3.102}$$

and

$$E - E_0 \simeq \frac{2\pi i}{3} - \left(-\frac{13}{15} + 2k\right)\frac{2\pi}{N}\sin\frac{10}{3}u + i(1-2k)\frac{2\pi}{N}\cos\frac{10}{3}u \tag{3.103}$$

Using the minimal value $k = 1$ and comparing with (1.25), we obtain

$$x_4 = 17/15 \tag{3.104}$$

with spin $s = 1$. The complex conjugate eigenvalues with zeros u_1 and u_2 in the lower half-plane is given by $x_5 = 17/15$ and $s = -1$. Notice that (3.103) gives a semitower on $x_1 = x_2 = 2/15$.

3.5. The Scaling Dimension $x = 4/3$

The eigenvalues $T(u)$ calculated in this subsection are characterized by four shifted zeros. Two of them, u_{1+} and u_{2+}, are located close to the lines $\mathrm{Re}(u) = -\pi/10$ and $\mathrm{Re}(u) = 3\pi/10$, from which they deviate toward the imaginary axis. The other two zeros, u_{1-} and u_{2-}, are located close to the lines $\mathrm{Re}(u) = -2\pi/10$ and $\mathrm{Re}(u) = 4\pi/10$, and deviate away from the imaginary axis. Here we will assume that u_{1+} and u_{2+} lie in the upper half-plane and u_{1-} and u_{2-} in the lower half-plane,

$$\begin{aligned} u_{1+} &= \frac{3i}{10}(y_{1+} + \ln N) - \frac{\pi}{10}, \qquad & u_{2+} &= \frac{3i}{10}(y_{2+} + \ln N) + \frac{3\pi}{10} \\ u_{1-} &= -\frac{3i}{10}(y_{1-} + \ln N) - \frac{2\pi}{10}, \qquad & u_{2-} &= -\frac{3i}{10}(y_{2-} + \ln N) + \frac{4\pi}{10} \end{aligned} \tag{3.105}$$

where y_{1+}, y_{2+} and y_{1-}, y_{2-} are complex conjugate pairs as shown in Fig. 13.

Proceeding as in the last two subsections but now using the function

$$f(u) := \frac{z_1(u - u_{1-} + \pi/5)\, z_1(u - u_{2-})}{z_1(u - u_{1+} - \pi/5)\, z_1(u - u_{2+} + 2\pi/5)} \tag{3.106}$$

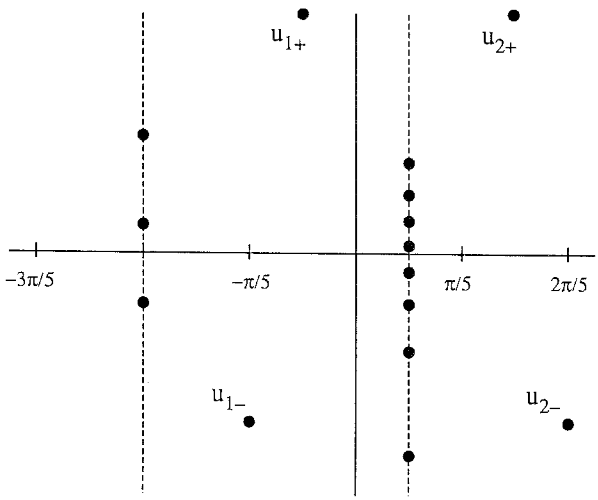

Fig. 13. Schematic representation of the zeros in the complex u plane of the second magnetic eigenvalue of critical hard hexagons. The finite-size correction to this eigenvalue yields the result $x = 4/3$.

we derive (3.52) or (3.80), where lf and lg are defined as before and the branches are specified by requiring

$$lf(\infty) = lg(\infty) = 0 \tag{3.107}$$

The asymptotics of these eigenvalues is given by

$$\ln \mathfrak{a}(\infty) = \ln[(\sqrt{5} - 1)/2], \qquad \ln \mathfrak{b}(\infty) = \ln[(\sqrt{5} - 1)/2] \tag{3.108}$$

$$\ln \mathfrak{A}(\pm\infty) = \ln[(1 + \sqrt{5})/2], \qquad \ln \mathfrak{B}(\pm\infty) = \ln[(1 + \sqrt{5})/2]$$
$$\lim_{N\to\infty} \ln \mathfrak{A}(0) = 0, \qquad \lim_{N\to\infty} \ln \mathfrak{B}(0) = 0 \tag{3.109}$$

From this we deduce

$$D_1 = D_2 = 0 \tag{3.110}$$

We next define limiting functions of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{A}$, and $\mathfrak{B}$ in the positive and negative scaling regimes and fix branches such that

$$la_\pm(\infty) = \ln[(\sqrt{5} - 1)/2], \qquad lb_\pm(\infty) = \ln[(\sqrt{5} - 1)/2]$$
$$lA_\pm(\infty) = \ln[(1 + \sqrt{5})/2], \qquad lB_\pm(\infty) = \ln[(1 + \sqrt{5})/2] \tag{3.111}$$
$$lA_\pm(-\infty) = 0, \qquad lB_\pm(-\infty) = 0$$

From (3.52), (3.80) we obtain the integral equations

$$\begin{aligned} la_\pm(x) &= -\sqrt{3}\, e^{-x} - lf_\pm(x) - s * lA_\pm - c * lB_\pm - 4\pi i/3 \\ lb_\pm(x) &= -\sqrt{3}\, e^{-x} - lg_\pm(x) - c * lA_\pm - s * lB_\pm + 4\pi i/3 \end{aligned} \tag{3.112}$$

where $lf_\pm$ and $lg_\pm$ are the scaling limits of lf and lg apart from trivial constants,

$$\begin{aligned} lf_\pm(x) &= -\ln\left[z_1\left(\frac{3i}{10}(x-y_{1\pm})-\frac{\pi}{4}\right) z_1\left(\frac{3i}{10}(x-y_{2\pm})-\frac{\pi}{20}\right)\right] \\ lg_\pm(x) &= +\ln\left[z_1\left(\frac{3i}{10}(x-y_{1\pm})+\frac{\pi}{20}\right) z_1\left(\frac{3i}{10}(x-y_{2\pm})+\frac{\pi}{4}\right)\right] \end{aligned} \tag{3.113}$$

The branches here are specified by requiring

$$lf_\pm(\infty) = -\frac{4\pi i}{3}, \qquad lg_\pm(\infty) = \frac{4\pi i}{3} \tag{3.114}$$

The finite-size corrections of T_1 are expressed in terms of $lA_\pm$ and $lB_\pm$ functions as follows:

$$\begin{aligned} \ln T_1(u) &\simeq N \ln z_1(u) + \frac{2\pi i}{3} \\ &\quad + \frac{e^x}{N}\left\{\frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-y}[lA_+(y) + lB_+(y)] \right. \\ &\quad \left. -\sqrt{3}(e^{\pi i/3 - y_{1+}} + e^{-\pi i/3 - y_{2+}})\right\} \\ &\quad + \frac{e^{-x}}{N}\left\{\frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-y}[lA_-(y) + lB_-(y)] \right. \\ &\quad \left. +\sqrt{3}(e^{-2\pi i/3 - y_{1-}} + e^{2\pi i/3 - y_{2-}})\right\} \end{aligned} \tag{3.115}$$

Instead of (3.96) and (3.100), we get

$$\begin{aligned} &-\sqrt{3}(e^{\pi i/3 - y_{1+}} + e^{-\pi i/3 - y_{2+}}) \\ &\quad = -(2k_+ - 1)2\pi + 8\pi/3 + i(c * lA_+ + s * lB_+)(y_{1+} + \pi i/6) \\ &\qquad - i(s * lA_+ + c * lB_+)(y_{2+} - \pi i/6) \\ &\qquad - i[lf_+(y_{2+} - \pi i/6) - lg_+(y_{1+} + \pi i/6)] \end{aligned} \tag{3.116}$$

where the last line is equal to -2π and

$$\begin{aligned}
&\frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-y}[lA_+(y)+lB_+(y)] \\
&\quad =\frac{1}{4\pi}\int_{-\infty}^{\infty} dx\, [la'_+(x)\, lA_+(x)-la_+(x)\, lA'_+(x) \\
&\qquad +lb'_+(x)\, lB_+(x)-lb_+(x)\, lB'_+(x)] \\
&\qquad -i(c*lA_+ +s*lB_+)\left(y_{1+}+\frac{\pi i}{6}\right) \\
&\qquad +i(s*lA_+ +c*lB_+)\left(y_{2+}-\frac{\pi i}{6}\right)
\end{aligned} \tag{3.117}$$

Similarly, instead of (3.68) and (3.72), we obtain

$$\begin{aligned}
&\sqrt{3}(e^{-2\pi i/3-y_{1-}}+e^{2\pi i/3-y_{2-}}) \\
&\quad =-(2k_- -1)2\pi+8\pi/3+i(c*lA_- +s*lB_-)(y_{1-}+\pi i/6) \\
&\qquad -i\,(s*lA_- +c*lB_-)(y_{2-}-\pi i/6) \\
&\qquad -i[lf_-(y_{2-}-\pi i/6)-lg_-(y_{1-}+\pi i/6)]
\end{aligned} \tag{3.118}$$

where again the last line is equal to -2π and

$$\begin{aligned}
&\frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-y}[lA_-(y)+lB_-(y)] \\
&\quad =\frac{1}{4\pi}\int_{-\infty}^{\infty} dx\, [la'_-(x)\, lA_-(x)-la_-(x)\, lA'_-(x) \\
&\qquad +lb'_-(x)\, lB_-(x)-lb_-(x)\, lB'_-(x)] \\
&\qquad -i(c*lA_- +s*lB_-)\left(y_{1-}+\frac{\pi i}{6}\right) \\
&\qquad +i(s*lA_- +c*lB_-)\left(y_{2-}-\frac{\pi i}{6}\right)
\end{aligned} \tag{3.119}$$

Inserting Eqs. (3.116)–(3.119) into (3.115), we find that this simplifies to give

$$\ln T_1(u) \simeq N \ln z_1(u) + \frac{2\pi i}{3}$$

$$+ \frac{e^x}{N} \left\{ \frac{1}{4\pi} \int_{-\infty}^{\infty} dx \, [la'_+(x)\, lA_+(x) - la_+(x)\, lA'_+(x) \right.$$

$$\left. + lb'_+(x)\, lB_+(x) - lb_+(x)\, lB'_+(x)] - k_+ 4\pi + \frac{8\pi}{3} \right\}$$

$$+ \frac{e^{-x}}{N} \left\{ \frac{1}{4\pi} \int_{-\infty}^{\infty} dx \, [la'_-(x)\, lA_-(x) - la_-(x)\, lA'_-(x) \right.$$

$$\left. + lb'_-(x)\, lB_-(x) - lb_-(x)\, lB'_-(x)] - k_- 4\pi + \frac{8\pi}{3} \right\}$$

$$= N \ln z_1(u) + \frac{2\pi i}{3} + \frac{e^x}{N} \left(\frac{41}{15}\pi - 4\pi k_+ \right) + \frac{e^{-x}}{N} \left(\frac{41}{15}\pi - 4\pi k_- \right)$$

$$= N \ln z_1(u) + \frac{2\pi i}{3} + \frac{\cosh x}{N} \left(\frac{41}{15} - 2k_+ - 2k_- \right) 2\pi$$

$$+ \frac{\sinh x}{N} (2k_- - 2k_+) 2\pi \qquad (3.120)$$

where we have used the values of the integrals (3.25) in the last step and k_+ and k_- are integers. We thus obtain the result

$$E \simeq -N \ln z_1(u) - \frac{2\pi i}{3} - \left(-\frac{41}{15} + 2k_+ + 2k_- \right) \frac{2\pi}{N} \sin \frac{10}{3} u$$

$$- i(2k_+ - 2k_-) \frac{2\pi}{N} \cos \frac{10}{3} u \qquad (3.121)$$

and

$$E - E_0 \simeq -\frac{2\pi i}{3} - \left(-\frac{8}{3} + 2k_+ + 2k_- \right) \frac{2\pi}{N} \sin \frac{10}{3} u$$

$$- i(2k_+ - 2k_-) \frac{2\pi}{N} \cos \frac{10}{3} u \qquad (3.122)$$

Choosing the minimal values $k_+ = k_- = 1$, we obtain the scaling dimension

$$x_6 = 4/3 \qquad (3.123)$$

with zero spin. The complex conjugate eigenvalue is given by the same numbers $x_7 = 4/3$ and zero spin. We again point out that (3.122) gives a semitower on $x_6 = x_7 = 4/3$.

3.6. The Scaling Dimension $x = 9/5$

The eigenvalues $T(u)$ calculated in this subsection are also characterized by four shifted zeros and are very similar to those treated in the last subsection. The first difference, however, lies in the fact that the zeros u_{1-} and u_{2-} deviate from the lines $\mathrm{Re}(u) = -2\pi/10$ and $\mathrm{Re}(u) = 4\pi/10$ toward the imaginary axis. Second, the asymptotic behavior of the eigenvalues $T(u)$ is different,

$$\ln \mathfrak{a}(\infty) = \ln[(1+\sqrt{5})/2] + \pi i, \qquad \ln \mathfrak{b}(\infty) = \ln[(1+\sqrt{5})/2] - \pi i \tag{3.124}$$

$$\ln \mathfrak{A}(\pm\infty) = \ln[(\sqrt{5}-1)/2] + \pi i, \qquad \ln \mathfrak{B}(\pm\infty) = \ln[(\sqrt{5}-1)/2] - \pi i$$
$$\lim_{N\to\infty} \ln \mathfrak{A}(0) = 0 \qquad \lim_{N\to\infty} \ln \mathfrak{B}(0) = 0 \tag{3.125}$$

Repeating the steps of the last subsection, we deduce

$$D_1 = \frac{2\pi}{3} i, \qquad D_2 = -\frac{2\pi}{3} i \tag{3.126}$$

We define limiting functions of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{A}$, and $\mathfrak{B}$ in the positive and negative scaling regimes and fix branches such that

$$la_{\pm}(\infty) = \ln[(1+\sqrt{5})/2] + \pi i, \qquad lb_{\pm}(\infty) = \ln[(1+\sqrt{5})/2] - \pi i$$
$$lA_{\pm}(\infty) = \ln[(\sqrt{5}-1)/2] + \pi i, \qquad lB_{\pm}(\infty) = \ln[(\sqrt{5}-1)/2] - \pi i$$
$$lA_{\pm}(-\infty) = 0, \qquad lB_{\pm}(-\infty) = 0 \tag{3.127}$$

From (3.52) or (3.80) we obtain the integral equations

$$\begin{aligned} la_{\pm}(x) &= -\sqrt{3}\, e^{-x} - lf_{\pm}(x) - s * lA_{\pm} - c * lB_{\pm} - 2\pi i/3 \\ lb_{\pm}(x) &= -\sqrt{3}\, e^{-x} - lg_{\pm}(x) - c * lA_{\pm} - s * lB_{\pm} + 2\pi i/3 \end{aligned} \tag{3.128}$$

where $lf_{\pm}$ and $lg_{\pm}$ are defined in (3.113) and (3.114).

The finite-size corrections of T_1 are expressed in terms of $lA_{\pm}$ and $lB_{\pm}$ functions as in (3.115) without the lattice momentum $2\pi i/3$. In order to evaluate the integrals, we note the following relations:

$$
\begin{aligned}
&-\sqrt{3}(e^{\pi i/3-y_{1+}}+e^{-\pi i/3-y_{2+}})\\
&\quad=-(2k_+-1)2\pi+4\pi/3+i(c*lA_++s*lB_+)(y_{1+}+\pi i/6)\\
&\qquad-i(s*lA_++c*lB_+)(y_{2+}-\pi i/6)\\
&\qquad-i[lf_+(y_{2+}-\pi i/6)-lg_+(y_{1+}+\pi i/6)]
\end{aligned}
\tag{3.129}
$$

where the last line is equal to -2π and

$$
\begin{aligned}
&\frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty}dy\,e^{-y}[lA_+(y)+lB_+(y)]\\
&\quad=\frac{1}{4\pi}\int_{-\infty}^{\infty}dx\,[la'_+(x)\,lA_+(x)-la_+(x)\,lA'_+(x)\\
&\qquad+lb'_+(x)\,lB_+(x)-lb_+(x)\,lB'_+(x)]\\
&\qquad-\frac{\pi}{3}-i(c*lA_++s*lB_+)\left(y_{1+}+\frac{\pi i}{6}\right)\\
&\qquad+i(s*lA_++c*lB_+)\left(y_{2+}-\frac{\pi i}{6}\right)
\end{aligned}
\tag{3.130}
$$

as well as

$$
\begin{aligned}
&\sqrt{3}\,(e^{-2\pi i/3-y_{1-}}+e^{2\pi i/3-y_{2-}})\\
&\quad=-(2k_--1)2\pi+4\pi/3+i(c*lA_-+s*lB_-)(y_{1-}+\pi i/6)\\
&\qquad-i(s*lA_-+c*lB_-)(y_{2-}-\pi i/6)\\
&\qquad-i[lf_-(y_{2-}-\pi i/6)-lg_-(y_{1-}+\pi i/6)]
\end{aligned}
\tag{3.131}
$$

where again the last line is equal to -4π and

$$
\begin{aligned}
&\frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty}dy\,e^{-y}[lA_-(y)+lB_-(y)]\\
&\quad=\frac{1}{4\pi}\int_{-\infty}^{\infty}dx\,[la'_-(x)\,lA_-(x)-la_-(x)\,lA'_-(x)\\
&\qquad+lb'_-(x)\,lB_-(x)-lb_-(x)\,lB'_-(x)]\\
&\qquad-\frac{\pi}{3}-i(c*lA_-+s*lB_-)\left(y_{1-}+\frac{\pi i}{6}\right)\\
&\qquad+i(s*lA_-+c*lB_-)\left(y_{2-}-\frac{\pi i}{6}\right)
\end{aligned}
\tag{3.132}
$$

Inserting (3.129), (3.130) and (3.131), (3.132) into (3.115) and ignoring the lattice momentum $2\pi i/3$, we find that this simplifies to give

$$\begin{aligned}
\ln T_1(u) &\simeq N \ln z_1(u) \\
&\quad + \frac{e^x}{N}\left\{\frac{1}{4\pi}\int_{-\infty}^{\infty} dx\,[la'_+(x)\,lA_+(x) - la_+(x)\,lA'_+(x) \right. \\
&\quad \left. + lb'_+(x)\,lB_+(x) - lb_+(x)\,lB'_+(x)] - k_+4\pi + \pi\right\} \\
&\quad + \frac{e^{-x}}{N}\left\{\frac{1}{4\pi}\int_{-\infty}^{\infty} dx\,[la'_-(x)\,lA_-(x) - la_-(x)\,lA'_-(x) \right. \\
&\quad \left. + lb'_-(x)\,lB_-(x) - lb_-(x)\,lB'_-(x)] - k_-4\pi - \pi\right\} \\
&= N \ln z_1(u) + \frac{e^x}{N}\left(\frac{19}{15}\pi - 4\pi k_+\right) + \frac{e^{-x}}{N}\left(-\frac{11}{15}\pi - 4\pi k_-\right) \\
&= N \ln z_1(u) + \frac{\cosh x}{N}\left(\frac{4}{15} - 2k_+ - 2k_-\right)2\pi \\
&\quad + \frac{\sinh x}{N}(1 + 2k_- - 2k_+)2\pi
\end{aligned} \tag{3.133}$$

where we have used the values of the integrals in the last step and k_+ and k_- are integers. We thus obtain the results

$$\begin{aligned}
E &\simeq N \ln z_1(u) - \left(-\frac{4}{15} + 2k_+ + 2k_-\right)\frac{2\pi}{N}\sin\frac{10}{3}u \\
&\quad - i(-1 + 2k_+ - 2k_-)\frac{2\pi}{N}\cos\frac{10}{3}u
\end{aligned} \tag{3.134}$$

and

$$\begin{aligned}
E - E_0 &\simeq -\left(-\frac{1}{5} + 2k_+ + 2k_-\right)\frac{2\pi}{N}\sin\frac{10}{3}u \\
&\quad - i(-1 + 2k_+ - 2k_-)\frac{2\pi}{N}\cos\frac{10}{3}u
\end{aligned} \tag{3.135}$$

Choosing the minimal values $k_+ = 1$, $k_- = 0$, we obtain the scaling dimension

$$x_8 = 9/5 \tag{3.136}$$

with spin $s = 1$. The complex conjugate eigenvalue is given by $x_9 = 9/5$ and spin $s = -1$. Notice here that (3.135) gives a semitower on $x_3 = 4/5$ and so completing the tower, except perhaps for degeneracies.

4. DISCUSSION

In this paper we have calculated the central charges and scaling dimensions of tricritical hard squares and critical hard hexagons from the finite-size corrections to the row transfer matrix eigenvalues. The calculated central charges $c=7/10$ for tricritical hard squares and $c=4/5$ for critical hard hexagons establish beyond doubt the generally accepted values. The various scaling dimensions calculated are summarized in (1.3). This is not an exhaustive list, but the methods presented here can be straightforwardly extended to analyze further excitations as required. All the results obtained are in complete agreement with the predictions of conformal invariance and modular invariance. More specifically, the scaling dimensions and spins agree with the Kac formula and the eigenvalue spectra are indeed found to be generated by the modular invariant partition functions (1.33) and (1.35). The indicated degeneracies of these levels have been confirmed for all levels corresponding to relevant scaling fields. Notice that excitations characterized by quite distinct patterns of zeros can in fact yield identical levels in the thermodynamic limit. In particular, we remark that it remains an open problem to completely classify the patterns of zeros that occur.

Of course much remains to be done. In this paper we have restricted ourselves to periodic boundary conditions with $N=0$ (mod 2 or mod 3) as appropriate. It would be of interest to obtain the eigenvalue spectra with these conditions relaxed to confirm the predicted effects of other boundary conditions. It is also naturally desirable to extend our results to the complete family of RSOS models introduced by Andrews *et al.*[42] and to other *A-D-E* models.[40,41] We hope to treat these models in future publications. There is also much interest in the spectra of quantum spin chains. The central charges of the spin-1/2 and spin-1 *XXZ* chains considered previously[25] can now be obtained analytically even with twisted boundary conditions by an extension of the methods of this paper. The details of this calculation will be published elsewhere.[43]

APPENDIX: ROGERS DILOGARITHMS

In this Appendix we collect together some basic properties and special values of Rogers dilogarithms.[44] The Rogers dilogarithm is defined by

$$\begin{aligned} L(x) &= -\frac{1}{2}\int_0^x dy\left[\frac{\ln(1-y)}{y}+\frac{\ln y}{1-y}\right] \\ &= -\int_0^x dy\,\frac{\ln(1-y)}{y}+\frac{1}{2}\ln x\ln(1-x) \end{aligned} \tag{A.1}$$

In particular,

$$L(0)=0, \qquad L(1)=-\int_0^1 dy\,\frac{\ln(1-y)}{y}=\sum_{n=0}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6} \tag{A.2}$$

By differentiating, it can be verified that $L(x)$ satisfies the simple functional equation

$$L(x)+L(1-x)=L(1)=\pi^2/6 \tag{A.3}$$

Similarly, $L(x)$ satisfies the two-variable functional relation

$$L(x)+L(y)=L(xy)+L\left(\frac{x(1-y)}{1-xy}\right)+L\left(\frac{y(1-x)}{1-xy}\right) \tag{A.4}$$

Setting $x=y=(\sqrt{5}-1)/2$ in these two linear equations and solving, it follows immediately that

$$L\left(\frac{\sqrt{5}-1}{2}\right)=\frac{\pi^2}{10}, \qquad L\left(\frac{3-\sqrt{5}}{2}\right)=\frac{\pi^2}{15} \tag{A.5}$$

In addition, from (A.3) with $x=1/2$, we obtain

$$L\left(\frac{1}{2}\right)=\frac{\pi^2}{12} \tag{A.6}$$

The dilogarithm

$$L_+(x)=\frac{1}{2}\int_0^x da\left[\frac{\ln(1+a)}{a}-\frac{\ln a}{1+a}\right]=L\left(\frac{x}{1+x}\right) \tag{A.7}$$

which occurs often in this paper, is simply related to the Rogers dilogarithm by changing the variable of integration to $y=a/(1+a)$. The definition of the Rogers dilogarithm (A.1) and the associated integral (A.7) can be extended[44] to complex values of x. The dilogarithms are then in general complex with branch cuts along the real axis.

ACKNOWLEDGMENTS

This work is supported by a grant from the Australian Research Council. A.K. acknowledges a Fellowship from Deutsche Forschungsgemeinschaft (DFG).

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